4.1 DISTANCE IN GRAPHS

1. Find the eccentricities of each vertex of the tree T of Figure 4.5. Do the same for the tree of Figure 4.7.

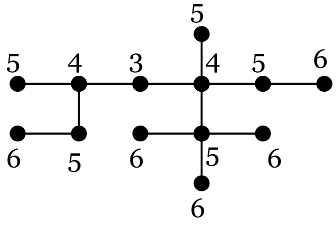


Figure 4.5

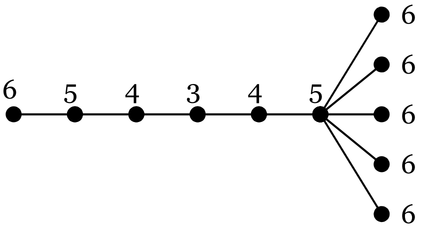


Figure 4.7

10. Prove that all trees have properties P4.1 – P4.4 earlier in this section.

P4.1 Given three vertices u, v, and w of a tree, such that u and v are adjacent, we have $|d(u, w) - d(w, v)| = 1$. In other words, one of u and v is closer to w by precisely one edge.

Proof: Let T be a tree and let u, v, and w be vertices of T such that u and v are adjacent. Without loss of generality, we will assume the distance from v to w is shorter than the distance from u to w. Note that the $d(u, w) \neq d(v, w)$ because this would imply that there are two separate paths – one from u to w and one from v to w – and that there is a cycle in T. Let P be the geodesic from v to w with $d(v, w) = k$. So, there is a path from u to w, goes from u to v and through path P. By P3.3, this uw path is unique. Thus, $d(u, v) = k + 1$ and $|d(u, w) - d(w, v)| = 1$.

P4.2 All eccentric vertices of a tree are end vertices.

Proof: Let T be a tree and let u and v be vertices of T such that v is an eccentric vertex of u. We want to show that it must be an end vertex. Suppose v is not an end vertex, then it is adjacent to some vertex w in T. Vertex w is not adjacent to v and the only path from w to u must go through v (P3.3). It follows that $e(u) \geq d(u, w) > d(u, v)$. This is contradicts the fact that v is an eccentric vertex of u. Therefore, v must be an end vertex.

P4.3 Pairs of antipodal vertices of a tree are end vertices.

Proof: Let T be a tree and let u and v be antipodal vertices of T. Then u and v are be mutually eccentric. By P4.2, u and v are end vertices.

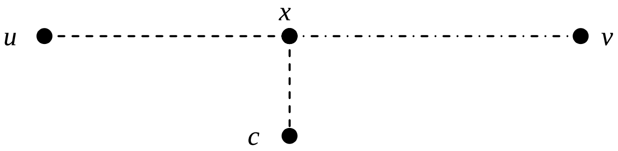
P4.4 The periphery of a tree consists of end vertices.

Proof: Let T be a tree and u be in the periphery of T. There exists a vertex v in T such that $d(u, v) = diam(T)$. Thus, u and v are mutually eccentric. By P4.2, all eccentric vertices are end vertices. Therefore u is an end vertex and $P(T)$ consists of end vertices.

11. Prove property P4.5: In any tree T, every diametral path includes all central vertices.

Proof: Let T be a tree and P be a diametral path from u to v in T. Let c be an arbitrary central vertex of T. We need to prove that the path P includes vertex c.

Suppose c in not in $V(P)$. Since trees are connected graphs, there exists a path from c to some vertex x in P. Note that x is not u or v because u and v are both end vertices of a path whose length is $diam(T)$.



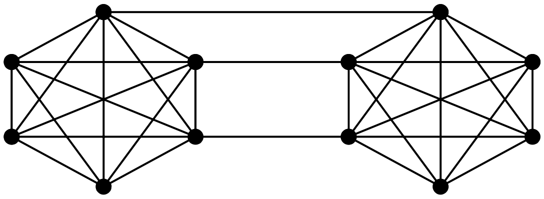
Since x is in the diametral path, $e(x) = \max\{d(u, x), d(x, v)\}$, and since c is a central vertex, we know that $e(c) \leq e(x)$. Thus, we have the following inequalities:

$$d(x, u) + d(x, c) \leq e(c) \leq e(x) = \max\{d(u, x), d(x, v)\}$$
$$d(x, v) + d(x, c) \leq e(c) \leq e(x) = \max\{d(u, x), d(x, v)\}$$

This implies that $d(x, c) = 0$, i.e. $x = c$, contradicting our assumption that c is not on the diametral path.

4.2 CONNECTIVITY CONCEPTS

9. Construct a graph G with $\kappa(G)=3$, $\lambda(G)=3$, and $\delta(G)=5$.



11. Determine $\kappa(G)$ and $\lambda(G)$ as a function of m and n for the generalized wheel $W_{m,n}=\overline{K}_m+C_n$.

$$\kappa(G)=\min\{n, m+2\}$$

Say we delete some vertices from C_n and some from $\overline{K}_m$. We would still have a path between any two vertices because they can go back and forth from $\overline{K}_m$ and C_n. This means that we will have to delete all the vertices of $\overline{K}_m$ or all the vertices of C_n. We know that deleting all the vertices from C_n will leave us with only $\overline{K}_m$ which is disconnected. If we delete all the vertices of $\overline{K}_m$, we still have a cycle. To disconnect a cycle, we have to delete two additional vertices. Thus, $\kappa(G)=\min\{n, m+2\}$. Note that the wheel $W_{m,n}=\overline{K}_m+C_n$ has minimum degree $\delta(G)=\min\{n, m+2\}$.

16. Prove that if diam(G) ≤ 2, then $\lambda(G)=\delta(G)$.

Let G be a graph such that $d(u, v) = diam(G) \leq 2$. Let $\delta(G)=k$ and suppose that there exists an edge cut set S with $|S| < k$. Let G_1 and G_2 be the components of $G - S$ and let G_1^* and G_2^* be the subsets of G_1 and G_2 respectively such that if $x \in G_1^*$ or G_2^* then x is incident to an edge in S.

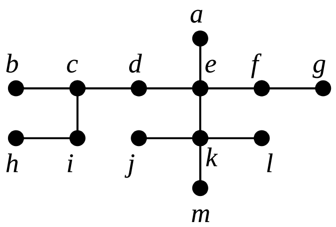
Take a vertex x in G_1^*. We claim that x must be adjacent to a vertex a in $G_1 - G_1^*$. Let s be the number of vertices in G_2^* adjacent to x through one of the edges in S. This means that $k - s$ edges in S are accounted for, leaving us with only $k - s$ edges left. Thus, there can be a maximum of $k - s$ other vertices in G_1^*. Counting the number of vertices in G_1^* and G_2^* adjacent to x, we have a maximum of $s + (k - s) = k$ possible adjacencies. This is less than the minimum degree of G. So, x must be adjacent to some vertex in $G_1 - G_1^*$. The same argument can be applied to the other component to obtain a vertex b in $G_2 - G_2^*$ adjacent to y.

Now, a and b are nonadjacent vertices because they are in different components of $G - S$ and $ab \notin S$. Furthermore, the shortest path from a to b distance 3, i.e. $a \to u \to v \to b$. We can conclude that the diameter of G must be at least 3. It follows (by the contrapositive) that $\lambda(G)=\delta(G)$.

4.3 APPLICATIONS

4. Must the induced subgraph of a central distance set be connected? If not, give an example.

No, see below.



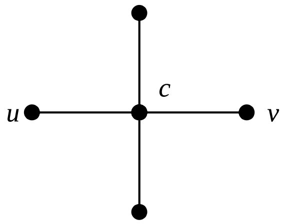
$C(G) = \{d\}$
$N_3(G) = \{g, h, j, l, m\}$

9. If G and H are F-graphs, determine when $G + H$ is an F-graph.

If G and H are F-graphs, then $G + H$ will be F-graph when $rad(G) = 1$ or $rad(H) = 1$ (or both). Note that the join of two graphs has a maximum distance of two between any two vertices, i. e. for all x, y in $G + H$, $d(x, y) \leq 2$. Thus, $rad(G + H) = 1$ or $rad(G + H) = 2$.

If $rad(G + H) = 1$, then there is a vertex u $u \in C(G + H)$ such that $d(u, x) = 1$ for all x in $V(G + H)$. Without loss of generality, let $u \in V(G)$. Vertex u is adjacent to every vertex in G. Since 1 is the shortest distance for any path in a connected simple graph, we can conclude that $u \in C(G)$. It is given that G is an F-graph. So, there is another vertex v in $V(G)$ with $\mathrm{d}(u, y) = 1$ for all y in $V(G)$ and $d(u, v) = 1$. Moreover, $e(v) = 1$ in $G + H$. Thus, $|C(G + H)| \geq 2$. It is obvious that the distance between any two vertices in $C(G + H)$ is 1.

11. Suppose that $G = K_{1,4}$ and u and v are two particular vertices at distance 2 from one another. Suppose further that the probability of edge failure is 0.2. Find the probability that u and v end up in the same component.



A= event that edge uc is missing
B = event that edge cv is missing
Pr(A) = Pr(B) = 0.2

Probability that u and v end up in the same component:
$1 - \Pr(A \cup B) = 1 - \Pr(A) - \Pr(B) + \Pr(A \cap B)$
$= 1 - 2(0.2) + (0.2)^2$
$= 0.64$