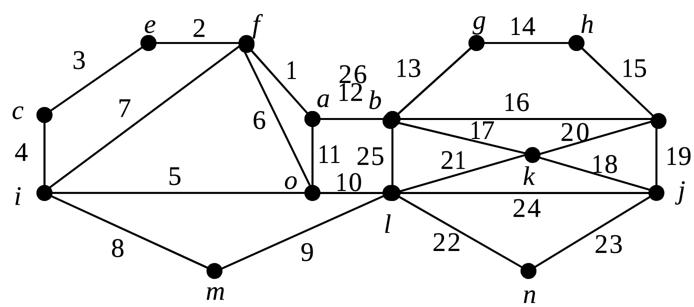
5.1 CHARACTERIZATION OF EULERIAN GRAPHS

9. The layout of streets in a certain city is displayed in Figure 5.10. The traffic department plans to travel along each street and paint new lines down the center of each street. Find a rout that can be used to begin at vertex *a* (the location of their depot), paint the lines on each street, and travel along as few extra streets as possible and return to the depot. Explain why we must travel along at one street twice, no matter how we design the route.

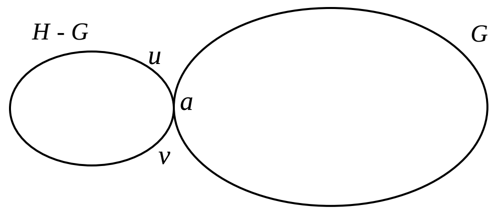
   Figure 5.10 is a semi-Eulerian graph – all vertices have even degree except for *a* and *b* which have odd degree. It follows that we must travel along one street twice. The order of the streets that the traffic department could travel along is given below.



11. Prove that a connected graph *G* is eulerian if and only if the edges of G can be partitioned into edge-disjoint cycles.

    Let *G* be an eulerian graph. It follows from the constructive technique in the proof of Theorem 5.1 that we can partition *G* into edge disjoint cycles.

    Conversely, suppose *G* is a connected graph such that *G* can be partitioned into edge-disjoint cycles. Let *n* the number of partitions of *G*. When $n = 1$, *G* is a circuit and hence eulerian. Assume that a connected graph *G* that can be partitioned into *n* edge disjoint cycles is eulerian. Now suppose we have a graph *H* that can be partitioned into $n + 1$ edge disjoint cycle. Look at the subgraph *G* of *H* where *G* contains *n* partitions of *H*. By our induction assumption, we can construct an eulerian circuit for *G*. Let *u* and *v* be in $H - G$ such that *u* and *v* are adjacent to the same vertex, say *a*, in the subgraph *G*. Such vertices exist because *H* is connected and $H - G$ is a cycle. The Eulerian circuit can be constructed as follows: $a \to u \to$ around cycle of $H - G \to v \to a \to$ around circuit of $G \to a$.



    Therefore, by induction, if the edges of a connected graph can be partitioned into disjoint cycles, then the graph is eulerian.

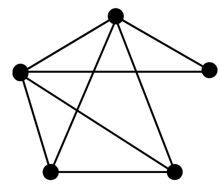
13. A knight (the one that looks like a horse) can move on a chessboard in an “L shape” either two spaces up or down and one to the side or two spaces to the side and one space up or down. Determine whether it is possible for a knight to tour an 8×8 chessboard making legal moves landing in each square exactly once and ending in the square it started.

It is possible. One such sequence of moves is given below.

34	51	32	15	38	53	18	3
31	14	35	52	17	2	39	54
50	33	16	29	56	37	4	19
13	30	49	36	1	20	55	40
48	63	28	9	44	57	22	5
27	12	45	64	21	8	41	58
62	47	10	25	60	43	6	23
11	26	61	46	7	24	59	42

5.2 HAMILTONICITY

7. Draw a connected graph having five vertices that is semi-hamiltonian and semi-eulerian.

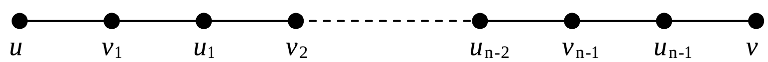


14. Prove that $K_{13,20}$ is not hamiltonian.

In a complete bipartite graph, the if *uv* is an edge, then *u* and *v* are in separate components. To trace out a hamiltonian circuit, we must go back and forth between the components. Since one component has 13 vertices and the other has 20 vertices, after going back and forth 13 times, we find that we must repeat some vertices in the first component. Thus, $K_{13,20}$ is not hamiltonian.

22. Prove that a

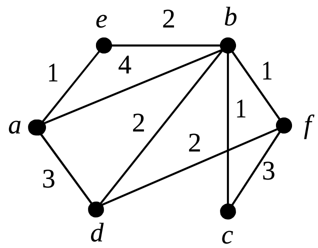
Let *G* be a maximal nonhamiltonian The addition of any edge to the graph of *G* will result in a hamiltonian graph. Let *u* and *v* be two nonadjacent vertices of *G*. Thus, $G + uv$ is hamiltonian cycle, and every hamiltonian cycle of $G + uv$ must contain edge *uv*. It follows that there is a path *P* from *u* to *v* in *G* that contains all the vertices of *G*, i.e. *P* is a spanning path of *G*.



By assumption, we do not have any hamiltonian cycles in G. So, for every vertex v_i adjacent to *u*, we have know vertex u_{i-1} is not adjacent to *v*. Excluding *u*, *v*, u_{n-1} and v_1, we have $2n - 4 = 2(n - 2)$ other vertices in the path. We can form $n - 2$ pairs with the other vertices. For each pairing, *u* and *v* cannot be adjacent to both vertices in any (u_i, v_{i+1}) pair. Counting edges, graph G can have at most $n^2 - 1 - (n - 2) = n^2 - n - 3$ edges, i.e a maximal nonhamiltonian bipartite graph on $2n$ vertices with equitable partitions *X* and *Y* can have at most $n^2 - n - 3$ edges. An graph with even one more edge will make it hamiltonian. Therefore, a bipartite graph *G* in which each part has order *n*, and *G* has at least $n^2 - n + 2$ edges, must be hamiltonian.

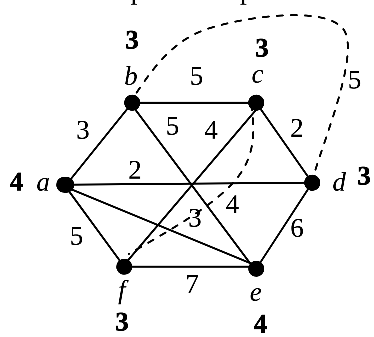
5.3 APPLICATIONS

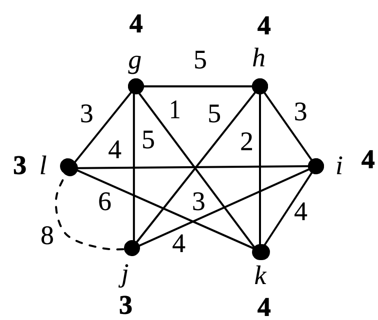
2. Solve the traveling salesman problem for the weighted graph in Figure 5.19.



There is only one hamiltonian cycle: *ebcfdae*
This cycle has a weight of 2 + 1 + 3 + 2 + 3 + 1 = 12.

4. Solve the Chinese postman problem for the graphs in Figure 5.22.

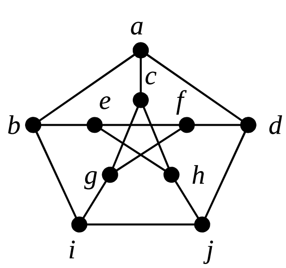


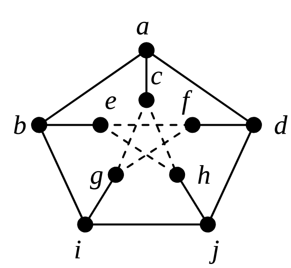


Eulerian tour: *abcdabeadefcfa*
Weight: 42 + 9 = 51

Eulerian tour: lgjhkihgklijl
Weight: 42 + 8 = 50

6. Prove that the Petersen graph is not hamiltonian.

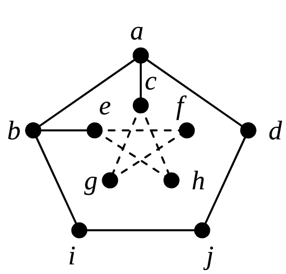


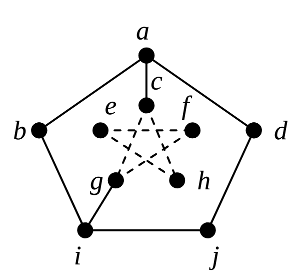


If a hamiltonian circuit exists, it will have 10 edges, with each vertex in the circuit having degree 2. We will examine the construction of such a hamiltonian circuit.

In the hamiltonian circuit, there will be at least one edge that starts at a vertex of the outer pentagon and ends at the a vertex in the inner star. Using the fact that there is symmetry in the Petersen graph, we can take this edge to be *ac*. We will need an even number of edges to connecting the inner star to the outer pentagon because we need to go in and back out.

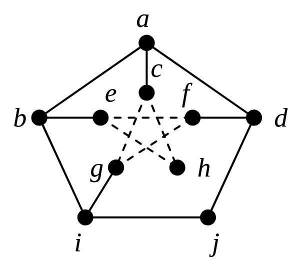
We have four cases for two edges from the pentagon to the star, but by symmetry, we can reduce it to two cases.





In the first graph, vertex *h* is skipped in the construction of a circuit; in the second graph we have to skip either vertex *b* or vertices *d* and *j* when constructing the circuit.

For the case with four edges from the outer pentagon to the inner star, we can pick any four edges. Again, symmetry reduces this to just one instance of this type.



There are 2 vertices of degree two in the above graph, vertex *h* and *j*. This means that we are forced to put edges *eh*, *hc*, *ij*, and *jd* in our circuit. This eliminates 5 vertices, *e*, *h*, *c*, *i*, *j*, and *d*. If we look at vertex *g*, we see that it is adjacent to vertices *c*, *f*, and *i*. Vertex g needs to have degree 2 in the hamiltonian circuit. But this is impossible since choosing any two adjacent vertices of *g* means that we will have to repeat a vertex.