6.1 #9 $\max\{diam(G) : G \text{ connected }, \beta(G) = k\} = 2k - 1$

Proof. Note that $diam(P_{2k}) = 2k - 1$ and $\beta(P_{2k}) = k$. Hence it suffices to show: if G is connected with $\beta(G) = k$ then $diam(G) \leq 2k - 1$. Suppose the contrary that $diam(G) \geq 2k$. Then there are vertices u, v such that $d(u, v) = 2k$. Let the shortest path between u and v be $u = u_0 - u_1 - u_2 - \cdots - u_{2k-1} - u_{2k} = v$. Hence the set $\{u_0, u_2, \ldots, u_{2k}\}$ is an independent set, otherwise there is a shorter path from u to v and so $2k = d(u, v) < 2k$, contradiction! Hence $\beta(G) \geq k + 1$, contradiction!!

6.2 #10 If G is k-regular connected with a cut-vertex then $\chi_1(G) = k + 1$.

Proof. Because of Vizing's Theorem, it suffices to show that $\chi_1(G) \neq k$. Suppose the contrary that $\chi_1(G) = k$. Hence $E(G)$ has a partition of k disjoint matchings $M_1 \cup \cdots \cup M_k$. The k-regularity and upper bound of matching size give

$$\frac{kn}{2} = |E(G)| = |M_1| + \cdots + |M_k| \leq \frac{n}{2} + \cdots + \frac{n}{2} = \frac{kn}{2}$$

and so $|M_i| = \frac{n}{2}$ for all i. It follows that n is even and G has k disjoint perfect matchings. Let v be a cut-vertex such that $G - v = G_1 \cup G_2$ where G_i are disjoint components in $G - v$. Take a neighbor a of v in G_1, and a neighbor b of v in G_2. Then there exists matchings M_1, M_2 such that the edge $av \in M_1$ and $bv \in M_2$. Note that the subgraph with edge set $M_1 \cup M_2$ is 2-regular, and so is a union of cycles. Hence av and bv belongs to the same cycle, and so there is a path from a to b in $G - v$, which is a contradiction because a and b belongs to two different components.