San Jose State University
Department of Mathematics, College of Science
Fall 2009
MATH 42, Discrete Mathematics
Answers of HW7

Please ask if you do not understand the answers.
Please report if you find any errors, typos.

4.1 # 4

a) $P(1) : 1^3 = \left(\frac{1(1+1)}{2}\right)^2$

b) LHS of $P(1) = 1^3 = 1$
RHS of $P(1) = \left(\frac{1(1+1)}{2}\right)^2 = 1^2 = 1$
Hence $P(1)$ is true

c) Assume that $P(k)$ is true for the positive integer $k \geq 1$ i.e. $1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$.

d) Prove that $P(k+1)$ is true i.e. $1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$.

e) Consider

$$\begin{aligned} 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= (1^3 + 2^3 + \cdots + k^3) + (k+1)^3 \\ &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\ &= (k+1)^2 \left(\frac{k^2}{4} + (k+1)\right) \\ &= (k+1)^2 \frac{k^2 + 4k + 4}{4} \\ &= \left(\frac{(k+1)(k+2)}{2}\right)^2 \end{aligned}$$

Hence $P(k+1)$ is true.

f) By the principle of Mathematical Induction, $P(n)$ is true for all positive integers greater than or equal to 1.

4.1 # 7

a) $P(0) : 3 \cdot 5^0 = \frac{3(5^{0+1}-1)}{4}$

b) LHS of $P(1) = 3 \cdot 5^0 = 3$
RHS of $P(1) = \frac{3(5^{0+1}-1)}{4} = 3$
Hence $P(1)$ is true

c) Assume that $P(k)$ is true for the nonnegative integer $k \geq 0$ i.e.
$3 + 3 \cdot 5 + \cdots + 3 \cdot 5^k = \frac{3(5^{k+1}-1)}{4}$.

d) Prove that $P(k+1)$ is true i.e.
$3 + 3 \cdot 5 + \cdots + 3 \cdot 5^k + 3 \cdot 5^{k+1} = \frac{3(5^{k+2}-1)}{4}$.

e) Consider

$$\begin{aligned} 3 + 3\cdot 5 + \cdots + 3\cdot 5^k + 3\cdot 5^{k+1} &= (3 + 3\cdot 5 + \cdots + 3\cdot 5^k) + 3\cdot 5^{k+1} \\ &= \frac{3(5^{k+1}-1)}{4} + 3\cdot 5^{k+1} \\ &= \frac{3}{4}\left(5^{k+1} - +4\cdot 5^{k+1}\right) \\ &= \frac{3(5^{k+2}-1)}{4} \end{aligned}$$

Hence $P(k+1)$ is true.

f) By the principle of Mathematical Induction, $P(n)$ is true for all nonnegative integers greater than or equal to 0.

4.1 # 14

a) $P(1) : 1\cdot 2^1 = (1-1)2^{1+1} + 2$

b) LHS of $P(1) = 1\cdot 2^1 = 2$
RHS of $P(1) = (1-1)2^{1+1} + 2 = 2$
Hence $P(1)$ is true

c) Assume that $P(k)$ is true for the positive integer $k \geq 1$ i.e.
$1\cdot 2^1 + 2\cdot 2^2 + \cdots + k\cdot 2^k = (k-1)2^{k+1} + 2$.

d) Prove that $P(k+1)$ is true i.e.
$1\cdot 2^1 + 2\cdot 2^2 + \cdots + k\cdot 2^k + (k+1)\cdot 2^{k+1} = ((k+1)-1)2^{(k+1)+1} + 2$.

e) Consider

$$\begin{aligned} 1\cdot 2^1 + 2\cdot 2^2 + \cdots + k\cdot 2^k + (k+1)\cdot 2^{k+1} &= (1\cdot 2^1 + 2\cdot 2^2 + \cdots + k\cdot 2^k) + (k+1)\cdot 2^{k+1} \\ &= (k-1)2^{k+1} + 2 + (k+1)\cdot 2^{k+1} \\ &= (k-1+k+1)2^{k+1} + 2 \\ &= k2^{k+2} + 2 \end{aligned}$$

Hence $P(k+1)$ is true.

f) By the principle of Mathematical Induction, $P(n)$ is true for all positive integers greater than or equal to 1.

4.1 # 19

a) $P(2) : 1 + \frac{1}{2^2} < 2 - \frac{1}{2}$

b) LHS of $P(2) = 1 + \frac{1}{2^2} = \frac{5}{4}$
RHS of $P(2) = 2 - \frac{1}{2} = \frac{3}{2}$
Hence $P(2)$ is true

c) Assume that $P(k)$ is true for the positive integer $k \geq 2$ i.e.
$1 + \frac{1}{2^2} + \cdots + \frac{1}{k^2} < 2 - \frac{1}{k}$.

d) Prove that $P(k+1)$ is true i.e.
$1 + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$.

e) Consider

$$\begin{aligned}
1+\frac{1}{2^2}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2} &= (1+\frac{1}{2^2}+\cdots+\frac{1}{k^2})+\frac{1}{(k+1)^2} \\
&< 2-\frac{1}{k}+\frac{1}{(k+1)^2} \\
&= 2-\frac{1}{k+1}+\frac{1}{k+1}-\frac{1}{k}+\frac{1}{(k+1)^2} \\
&= 2-\frac{1}{k+1}-\frac{1}{k(k+1)^2} \\
&< 2-\frac{1}{k+1}
\end{aligned}$$

Hence $P(k+1)$ is true.

f) By the principle of Mathematical Induction, $P(n)$ is true for all positive integers greater than or equal to 2.

4.1 # 22 a) $P(4) : 4^2 \leq 4!$

b) LHS of $P(4) = 4^2 = 16$
RHS of $P(4) = 4! = 24$
Hence $P(4)$ is true

c) Assume that $P(k)$ is true for the positive integer $k \geq 4$ i.e.
$k^2 \leq k!$.

d) Prove that $P(k+1)$ is true i.e.
$(k+1)^2 \leq (k+1)!$.

e) Consider

$$\begin{aligned}
(k+1)^2 &= k^2+2k+1 \\
&\leq k!+k!k \\
&= (k+1)!
\end{aligned}$$

Hence $P(k+1)$ is true.

f) By the principle of Mathematical Induction, $P(n)$ is true for all positive integers greater than or equal to 4.