Fall 2018
Answers of Chapter 5 homework

5.1: # 5.4 Proof: (counterexample)
\[ n = 2 \] is a counterexample because \( \frac{n(n+1)}{2} = 3 \) is odd but \( \frac{(n+1)(n+2)}{2} = 6 \) is even.

5.1: # 5.10 Proof: (counterexample)
\[ a = b = c = 2 \] is a counterexample because \( \{ab, ac, bc\} = \{4, 4, 4\} \) and so all of them are even, NO two of them are of opposite parity.

5.2: # 5.22 Proof: (proof by contradiction)
Assume the contrary that \( \sqrt{2} + \sqrt{3} \) is rational. Then \( \frac{1}{2}[(\sqrt{2} + \sqrt{3})^2 - 2 - 3] = \sqrt{6} \) is also rational. Now write \( \sqrt{6} = \frac{a}{b} \) where \( a \) and \( b \) has NO common divisor other than 1. Hence \( a^2 = 6b^2 \), and so \( 2|a^2 \). It follows that \( a = 2k \) is even. Therefore \( 2k^2 = 3b^2 \), and so \( 2|3b^2 \). It follows that \( b = 2h \) is even. Consequently, \( a \) and \( b \) have a common divisor 2, contradiction!

5.2: # 5.28 Proof: (proof by contradiction)
Assume the contrary that there are two odd integers \( a \) and \( b \) such that \( a^2 + b^2 = c^2 \) for some integer \( c \). Then \( a^2 \equiv 1 \pmod{4} \), and \( b^2 \equiv 1 \pmod{4} \). On the other hand, \( c^2 \equiv 0, 1 \pmod{4} \). Consequently \( 2 \equiv a^2 + b^2 = c^2 \equiv 0, 1 \pmod{4} \), contradiction.

5.3: # 5.36 Proof:
(a) (direct proof)
Since \( n \) is odd, \( n = 2k + 1 \) for some \( k \in \mathbb{Z} \). Then \( 7n - 5 = 2(7k + 1) \) is even because \( 7k + 1 \in \mathbb{Z} \).
(b) (proof by contrapositive)
Since \( 7n - 5 \) is odd, \( 7n - 5 = 2k + 1 \) for some \( k \in \mathbb{Z} \). Then \( n = (7n - 5) - 6n + 5 = 2k + 1 - 6n + 5 = 2(k - 3n + 3) \) is even because \( k - 3n + 3 \in \mathbb{Z} \).
(c) (proof by contradiction)
Assume the contrary that \( 7n - 5 \) is odd. Moreover \( n \) is odd. Then \( 7n - 5 = 2k + 1 \) for some \( k \in \mathbb{Z} \), and \( n = 2h + 1 \) for some \( h \in \mathbb{Z} \). Hence \( 2k + 1 = 7(2h + 1) - 5 \) and so \( 1 = 2(k - 7h) \) is even, contradiction.

5.3: # 5.39 Proof:
(a) (direct proof)
Since \( 0 < x \leq y \), we have \( x \cdot x \leq y \cdot y \), i.e., \( x^2 \leq y^2 \).
(b) (proof by contrapositive)
contrapositive: \( x^2 > y^2 \Rightarrow x > y \)
Since \( y^2 < x^2 \), we have \( (x+y)(x-y) = x^2 - y^2 > 0 \). Hence \( (x-y) = \frac{1}{x+y} \cdot (x+y)(x-y) > 0 \), i.e., \( x > y \).
(c) (proof by contradiction)
Assume the contrary that \( x^2 > y^2 \). As in part (b), we deduce that \( x > y \), contradicting \( x \leq y \).

5.4: # 5.44 Proof:
Take \( a = 2\sqrt{2} \neq \sqrt{2} = b \). Then both are irrational, and \( a^b = 2^{\sqrt{2}} \sqrt{2} = 4 \) is rational.
5.4: # 5.46  
Proof:  
(existence) Consider the continuous function $f(x) = x^3 + x^2 - 1$. Note that $f\left(\frac{2}{3}\right) = \frac{7}{27} < 0$ and $f(1) = 1 > 0$. Now, by Intermediate Value Theorem, there is $c \in \left(\frac{2}{3}, 1\right)$ such that $c^3 + c^2 - 1 = f(c) = 0$. 

(uniqueness) Suppose that there exist $\frac{2}{3} < a < b < 1$ such that $a^3 + a^2 - 1 = 0 = b^3 + b^2 - 1$. Then $(b - a)(b^2 + ab + a^2 + b + a) = b^3 - a^3 + b^2 - a^2 = 0$, and so $b - a = 0$.

5.5: # 5.56  
Proof:  
(proof by contradiction)  
Assume the contrary that there exists $x \in \mathbb{R}$ such that $x^6 + x^4 + 1 = 2x^2$. Then $x^6 + (x^2 - 1)^2 = 0$, and so $x^6 = 0$ and $(x^2 - 1)^2 = 0$. Hence $x = 0$ and $x = \pm 1$, contradiction. 

5.5: # 5.60  
(a) Take $p = 3$ and $q = 5$. Then $pq \pm 2, pq \pm 4$ are $11, 13, 17, 19$.
(b) (proof by contradiction)  
Assume the contrary that there exists primes $p$ and $q$ such that $pq \pm 2, pq \pm 4, pq \pm 6$ are all primes. By the Lemma below, one of $pq - 6, pq - 4, pq - 2$ is divisible by 3. Since they are primes, we have $pq - 6 = 3$, and so $pq + 6 = 15$, a contradiction because $pq + 6$ is a prime. 

Lemma: For any $n \in \mathbb{N}$, one of $n, n + 2, n + 4$ is divisible by 3. 
Proof: (proof by cases)  
Case 1: $n \equiv 0 \pmod{3}$. Then $n$ is divisible by 3.  
Case 2: $n \equiv 1 \pmod{3}$. Then $n + 2 \equiv 1 + 2 = 3 \equiv 0 \pmod{3}$, and so $n + 2$ is divisible by 3.  
Case 3: $n \equiv 2 \pmod{3}$. Then $n + 4 \equiv 2 + 4 = 6 \equiv 0 \pmod{3}$, and so $n + 4$ is divisible by 3.