6.1: # 6.6  (a) The sum $1^3 + 2^3 + \cdots + n^3$ is the number of cubes in an $n \times n \times n$ cube composed of $n^3$ $1 \times 1 \times 1$ cubes.

(b) (proof by induction)
Since $1^3 = 1 = \frac{1^2(1+1)^2}{4}$, the formula holds for $n = 1$. Assume that

$$1^3 + 2^3 + \cdots + n^3 = \frac{k^2(k+1)^2}{4}$$

for a positive $k$. We observe that

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$
$$= \frac{(k+1)^2}{4} \left(k^2 + 4(k+1)\right)$$
$$= \frac{(k+1)^2}{4} (k+2)^2$$

By principle of MI,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

for all positive integer $n$.

6.1: # 6.12  (a) Assume that

$$9 + 13 + \cdots + (4k + 5) = \frac{4(k^2 + 14(k) + 1}{2}$$

for a positive integer $k$. We observe that

$$9 + 13 + \cdots + (4k + 5) + (4k + 9) = \frac{4(k^2 + 14(k) + 1}{2} + (4k + 9)$$
$$= \frac{1}{2} \left(4(k^2 + 14(k) + 1 + 8k + 18\right)$$
$$= \frac{4(k+1)^2 + 14(k+1) + 1}{2}$$

(b) No, $P(1)$ is false because

$$4(1) + 5 = 9 \neq \frac{19}{2} = \frac{4(1)^2 + 14(1) + 1}{2}$$

6.2: # 6.18  (proof by induction)
Since $2^{10} = 1024 > 1000 = 10^3$, the inequality holds for $n = 10$. Assume that $2^k > k^3$ for a positive integer $k \geq 10$. Observe that $k^3 \geq 10k^2 = 3k^2 + 3k^2 + 4k^2 \geq 3k^2 + 3k + 1$ and

$$2^{k+1} = 2^k + 2^k$$
$$> k^3 + k^3$$
$$\geq k^3 + 3k^2 + 3k + 1$$
$$= (k+1)^3$$

By Modified Principle of MI,

$$2^n > n^3$$

for all positive integer $n \geq 10$. 
6.2: # 6.26 (proof by induction)
Since $10^{k+1} - 9(0) - 10 = 0$, the divisibility holds for $n = 0$. Assume that $81|(10^{k+1} - 9k - 10)$.
Then, by definition, $10^{k+1} - 9k - 10 = 81m$ for some $m \in \mathbb{Z}$. Observe that
\[
10^{k+2} - 9(k + 1) - 10 = 10(10^{k+1}) - 9k - 19 \\
= 10(10^{k+1} - 9k - 10) + 81k + 81 \\
= 81(m + k + 1)
\]
hence $81|(10^{k+2} - 9(k + 1) - 10)$ because $m + k + 1 \in \mathbb{Z}$. By Modified Principle of MI,
\[
81|(10^{n+1} - 9n - 10)
\]
for all positive integer $n \geq 0$.

6.3: # 6.34 Conjecture: $a_n = 2^{n-1}$ for $n \geq 1$.
Proof: (proof by strong induction)
Since $a_1 = 1 = 2^{1-1}$ and $a_2 = 2 = 2^{2-1}$, the statement is true for $n = 1, 2$. Assume that $a_n = 2^{n-1}$ for all $n = 1, 2, \ldots, k$ where $k \geq 2$. Observe that
\[
a_{k+1} = a_k + 2a_{k-1} = 2^{k-1} + 2 \cdot 2^{k-1-1} = 2^k = 2^{(k+1)-1}.
\]
By Strong Principle of MI, $a_n = 2^{n-1}$ for all positive integers $n \geq 1$.

6.3: # 6.36 (a) $F_1 = 1, F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$
(b) Let $P(n)$ be the statement that $2|F_n \iff 3|n$. We prove that $P(n)$ is true for all $n \geq 1$
by strong induction as follows.

Since $F_1 = 1, 2 \nmid 1$ and $3 \nmid 1$, we have $P(1)$ is true. Since $F_2 = 1, 2 \nmid 1$ and $3 \nmid 2$, we have
$P(2)$ is true. Assume that $P(i)$ is true for all $i = 1, 2, \ldots, k$ with $k \geq 2$. Observe that
\[
2|F_{k+1} \iff 2|F_k + (F_k - 1) \\
\iff (2|F_k \land 2|F_{k-1}) \lor (2 \nmid F_k \land 2 \nmid F_{k-1}) \\
\iff (3k \land 3|(k-1)) \lor (3 \nmid k \land 3 \nmid (k-1)) \\
\iff F \lor (3|(k+1)) \\
\iff 3|(k+1)
\]
By Strong Principle of MI, $2|F_n \iff 3|n$ for all positive integers $n \geq 1$.

6.4: # 6.40 (proof by minimum counterexample)
Assume the contrary that there is a minimum $n_0 \in \mathbb{N}$ such that $r^{n_0} + \frac{1}{r^{n_0}} \notin \mathbb{Z}$. Hence
\[
r^n + \frac{1}{r^n} \in \mathbb{Z}
\]
for $n = 1, 2, \ldots, n_0 - 1$, and $n_0 \geq 2$ because $r + \frac{1}{r} \in \mathbb{Z}$. Note that
\[
r^{n_0} + \frac{1}{r^{n_0}} = (r^{n_0-1} + \frac{1}{r^{n_0-1}})(r + \frac{1}{r}) - (r^{n_0-2} + \frac{1}{r^{n_0-2}}) \in \mathbb{Z}
\]
It contradicts that $r^{n_0} + \frac{1}{r^{n_0}} \notin \mathbb{Z}$.

6.4: # 6.42 (proof by minimum counterexample)
Assume the contrary that there is a minimum $n_0 \in \mathbb{N}$ such that $5 \nmid (n_0^5 - n_0)$ for $n = 1, 2, \ldots, n_0 - 1$. Note that
\[
(n_0^5 - n_0) = [(n_0 - 1)^5 - (n_0 - 1)] + 5([n_0 - 1] + 2(n_0 - 1)^3 + 2(n_0 - 1)^2 + (n_0 - 1))
\]
and so $5|(n_0^5 - n_0)$ is divisible by 5 because $5|[[(n_0 - 1)^5 - (n_0 - 1)]$. Hence it contradicts that
\[
5 \nmid (n_0^5 - n_0).
\]