8.3: # 8.36  proof by example

Proof: 11 is an odd integer whose digit-sum is 2 (even) and whose digit-product is 1 (odd).

8.3: # 8.48  direct proof

Proof: Let \( n \) be an even integer. then \( n = 2k \) for some \( k \in \mathbb{Z} \). Hence \( n = 2k = 2(k - 1) + 2(1) \) is the sum of two even integers because \( k - 1, 1 \in \mathbb{Z} \).

8.3: # 8.52  proof by counter-example

Disproof: Take \( a = 3 \) and \( c = 1 \), then there is NO positive \( b \) such that \( a + b = 3 + b = 1 = c \) because \( b = c - a = 1 - 3 = -2 \).

8.3: # 8.56  proof by contradiction

Disproof: Assume the contrary that there exists \( x \in \mathbb{R} \) such that \( x^2 < x < x^3 \). Note that \( x \neq 0 \) because \( 0^2 = 0 = 0^3 \). Hence \( x^2 > 0 \), and so \( x^2 < x^3 \) implies that \( 1 < x \). Consequently, \( x < x^2 \), contradicting that \( x^2 < x \).

8.3: # 8.60  proof by contrapositive

Proof: contrapositive: if \( A \neq \emptyset \) then there exists \( B \) such that \( A - B \neq \emptyset \).

Assume that \( A \neq \emptyset \). Now take \( B = \emptyset \) and so \( A - B = A \neq \emptyset \).

8.3: # 8.66  proof by counter-example

Disproof: Take \( A = \{1\}, B = \{2\} \). Then \( \mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \), but \( \mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}\} \).

8.3: # 8.70  proof by counter-example

Disproof: Take \( A = \{1\}, B = \{2\}, C = \{3\} \). Then \( A \cup (B - C) = \{1\} \), but \( (A \cup B) - (A \cup C) = \{2\} \).

8.3: # 8.76  proof by example

Proof: Take \( x = 51, y = 50 \). Then \( 51^2 - 50^2 = 101 \).

8.3: # 8.78  direct proof

Proof: Since \( p \) is odd and positive, \( p = 2k + 1 \) for some \( k \geq 1 \). Take \( a = k + 1 \) and \( b = k \) both positive integers, and \( p = 2k + 1 = (k + 1)^2 - k^2 = a^2 - b^2 \).

Remark: The proof does not use the fact that \( p \) is prime. Hence the conclusion is true for any odd positive integer greater 1.

8.3: # 8.80  direct proof

Proof: 5\( | \) (sum of 5 consecutive integers)

Sum of 5 consecutive integers \( = n + (n + 1) + (n + 2) + (n + 3) + (n + 4) = 5n + 10 = 5(n + 2) \) for some \( n \in \mathbb{Z} \). Hence 5\( | \) (sum of 5 consecutive integers) because \( n + 2 \in \mathbb{Z} \).

6 \( \not| \) (sum of 6 consecutive integers)

Sum of 6 consecutive integers \( = n + (n + 1) + (n + 2) + (n + 3) + (n + 4) + (n + 5) = 6n + 15 \) for some \( n \in \mathbb{Z} \). Hence Sum of 6 consecutive integers \( = 6(n + 2) + 3 \), and so 6 \( \not| \) (sum of 6 consecutive integers).