9.1: # 9.4 \[ R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, c)\} \]

9.1: # 9.6 \( cR^{-1}d \iff dRc \iff \frac{d}{c} \in \mathbb{N} \)

9.1: # 9.10 The maximum is 6. Let \( A = \{a, b, c, d\} \), then
\[ R = \{(a, b), (a, c), (a, d), (b, c), (b, d), (c, d)\} \]
is an example of cardinality 6 such that \( R \cap R^{-1} = \emptyset \). Moreover, we prove that if \( R \cap R^{-1} = \emptyset \) then \( |R| \leq 6 \).

Proof: Claim: \( \forall a \in A, (a, a) \notin R \). Assume the contrary that \( (a, a) \in R \) for some \( a \in A \). Then \( (a, a) \in R^{-1} \) by definition of \( R^{-1} \). Hence \( (a, a) \in R \cap R^{-1} \), contradicting that \( R \cap R^{-1} = \emptyset \). Let \( D = \{(a, a) : a \in A\} \). Then \( D \cap R = D \cap R^{-1} = \emptyset \). Note that \( D \cup R \cup R^{-1} \subseteq A \times A \), and so \( 4 + |R| + |R| = |D| + |R| + |R^{-1}| \leq |A \times A| = 16 \) because \( |D| = |A| = 4 \) and \( |R| = |R^{-1}| \). Consequently, \( |R| \leq 6 \).

9.2: # 9.12 (i) \( R \) is not reflexive b/c \((b, b) \notin R\) (ii) \( R \) is not symmetric b/c \((a, b) \notin R\) but \((b, a) \notin R\) (iii) \( R \circ R = R \) and hence \( R \) is transitive

9.2: # 9.18 (a) \( \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (2, 3), (3, 2)\} \)
(b) \( \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2)\} \)
(c) \( \{\}\) i.e. the empty relation
(d) \( \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3)\} \)
(e) \( \{(1, 2), (2, 1)\} \)
(f) \( \{(1, 2)\} \)

9.2: # 9.20 The maximum is 7. Let the set be \( A = \{a, b, c\} \), then
\[ R = \{(a, b), (a, c), (b, a), (b, b), (b, c), (c, b), (c, c)\} \]
is an example of cardinality 7 such that it is not reflexive, not symmetric, and not transitive. Moreover, we prove that if \( R \) is not reflexive, not symmetric, and not transitive then \( |R| \leq 7 \).

Proof: contrapositive: if \( |R| > 7 \) then \( R \) is reflexive, symmetric or transitive
Since \( 7 < |R| \leq 9 \), \( |R| = 8 \) or 9. If \( |R| = 9 \) then \( R = A \times A \) which is reflexive, symmetric and transitive, done. If \( |R| = 8 \) then \( R \) has an element less than \( A \times A \).
Case 1: the missing element is \((x, x)\) for some \( x \in A \). Then \( R \) is symmetric.
Case 2: the missing element is \((x, y)\) for some \( x, y \in A \) and \( x \neq y \). Then \( R \) is reflexive.

9.3: # 9.24 The equivalence classes are \( \{a, c, d, g\}, \{b, f\} \) and \( \{e\} \).
\[ R = \{(a, a), (a, c), (c, a)(a, d), (d, a), (a, g), (g, a), (c, c), (c, d), (d, c), (c, g), (g, c), (d, d), (d, g), (g, d), (g, g), (b, b), (b, f), (f, b), (f, f), (e, e)\} \]

9.3: # 9.30 (a) For \( a \in \mathbb{Q}^{+}, \frac{a}{2} = 1 = 2^{0} \), and so \((a, a) \in R\). Hence \( R \) is reflexive.
If \((a, b) \in R\) then \( \frac{b}{a} = 2^{m}\) for some \( m \in \mathbb{Z} \). Therefore, \( \frac{b}{a} = 2^{-m}\), and so \((b, a) \in R\) because \(-m \in \mathbb{Z}\). Hence \( R \) is symmetric.
If \((a, b), (b, c) \in R\) then \( \frac{b}{a} = 2^{m}\) and \( \frac{b}{c} = 2^{n}\) for some \( m, n \in \mathbb{Z} \). Therefore \( \frac{b}{c} = \frac{a}{b} \cdot \frac{b}{c} = 2^{m+n}\), and so \((a, c) \in R\) because \( m + n \in \mathbb{Z}\). Hence \( R \) is transitive.
Consequently, \( R \) is an equivalence relation.
9.3: # 9.34 (a) Since $R$ is reflexive, $(1, 1) \in R$. By definition, $0 = 1 - 1 \in H$.

(b) If $a \in H$, then $a - 0 \in H$, and so $(a, 0) \in R$. Since $R$ is symmetric, $(0, a) \in R$. By definition again, $-a = 0 - a \in H$.

(c) If $a, b \in H$ then $(a, 0), (0 - b) \in H$ by definition. Since $R$ is transitive, $(a, -b) \in R$, and so $a + b = a - (b) \in H$.

9.4: # 9.38 $R$ is not an equivalence relation because it is not transitive: $(4, 6), (6, 3) \in R$ but $(4, 3) \notin R$.

9.4: # 9.40 Proof:
Since $3x - 7x = -4x$ is even, $(x, x) \in R$ for all $x \in \mathbb{Z}$, and so $R$ is reflexive. If $(x, y) \in R$ then $3x - 7y$ is even and so
$$3y - 7x = (3x - 7y) + 10(y - x)$$
is also even, hence $(y, x) \in R$, i.e., $R$ is symmetric. If $(x, y), (y, z) \in R$ then both $3x - 7y$ and $3y - 7z$ is even, and so
$$3x - 7z = (3x - 7y) + 4y + (3y - 7z)$$
is even, hence $(x, z) \in R$, i.e., $R$ is transitive.

Note that $(x, y) \in R$ iff $x$ and $y$ has same parity. Consequently, there are two distinct equivalence classes: the even integers and the odd integers.

9.4: # 9.42 disprove by example
Take $A = \{1, 2, 3\}$. Let $R = \{(1, 1), (2, 2)(2, 3), (3, 2), (3, 3)\}$ and $S = \{(1, 1), (2, 2)(1, 3), (3, 1), (3, 3)\}$. Then both $R$ and $S$ are equivalence relations on $A$, but the union
$$R \cup S = \{(1, 1), (2, 2)(2, 3), (1, 3), (3, 1), (3, 2), (3, 3)\}$$
is not an equivalence relation because it is not transitive: $(2, 3), (3, 1) \in R \cup S$ but $(2, 1) \notin R \cup S$.

9.5: # 9.44 (a) True because $25 - 9 = 16$ is a multiple of $8$.

(b) False because $-17 - 9 = -26$ is NOT a multiple of $8$

(c) True because $-14 - (-14) = 0$ is a multiple of $4$

(d) False because $25 - (-3) = 28$ is NOT a multiple of $11$

9.5: # 9.50 Proof:
For any $a \in \mathbb{Z}$, $2a + 2a = 4a \equiv 0 \pmod{4}$, and so $(a, a) \in R$. Hence $R$ is reflexive.
If $(a, b) \in R$ then $2a + 2b \equiv 0 \pmod{4}$, and so $2b + 2a = 2a + 2b \equiv 0 \pmod{4}$, i.e., $(b, a) \in R$. Hence $R$ is symmetric.
If $(a, b), (b, c) \in R$ then $2a + 2b \equiv 0 \pmod{4}$ and $2b + 2c \equiv 0 \pmod{4}$, and so $2a + 2c = 2a + 4b + 2c = 2a + 2b + 2b + 2c \equiv 0 + 0 = 0 \pmod{4}$, i.e., $(a, c) \in R$. Hence $R$ is transitive.
There are two distinct equivalence classes: $[0] =$ even integers and $[1] =$ odd integers.

9.5: # 9.52 Proof:
For any $a \in \mathbb{Z}$, $a^2 \equiv a^2 \pmod{5}$, and so $(a, a) \in R$. Hence $R$ is reflexive.
If $(a, b) \in R$ then $a^2 \equiv b^2 \pmod{5}$, and so $b^2 \equiv a^2 \pmod{5}$, i.e., $(b, a) \in R$. Hence $R$ is symmetric.
If $(a, b), (b, c) \in R$ then $a^2 \equiv b^2 \pmod{5}$ and $b^2 \equiv c^2 \pmod{5}$, and so $a^2 \equiv c^2 \pmod{5}$, i.e., $(a, c) \in R$. Hence $R$ is transitive.
There are three distinct equivalence classes: $[0] = \{5k : k \in \mathbb{Z}\}$, $[1] = \{5k + 1 : k \in \mathbb{Z}\} \cup \{5k + 4 : k \in \mathbb{Z}\}$, and $[2] = \{5k + 2 : k \in \mathbb{Z}\} \cup \{5k + 3 : k \in \mathbb{Z}\}$. 

\[(b) \ [3] = \{x : (x, 3) \in R\} = \{x : \frac{3}{7} = 2m \text{ for some } m \in \mathbb{Z}\} = \{3 \cdot 2^m : m \in \mathbb{Z}\}\]
9.6: # 9.54 For $\mathbb{Z}_4$, 
\[
\begin{array}{c|cccc}
+ & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 2 & 3 & 0 \\
2 & 2 & 3 & 0 & 1 \\
3 & 3 & 0 & 1 & 2 \\
\end{array}
\quad 
\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 3 \\
2 & 0 & 2 & 0 & 2 \\
3 & 0 & 3 & 2 & 1 \\
\end{array}
\]

For $\mathbb{Z}_5$, 
\[
\begin{array}{c|ccccc}
+ & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 1 & 2 & 3 & 4 \\
1 & 1 & 2 & 3 & 4 & 0 \\
2 & 2 & 3 & 4 & 0 & 1 \\
3 & 3 & 4 & 0 & 1 & 2 \\
4 & 4 & 0 & 1 & 2 & 3 \\
\end{array}
\quad 
\begin{array}{c|ccccc}
\cdot & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 3 & 4 \\
2 & 0 & 2 & 4 & 2 & 3 \\
3 & 0 & 3 & 1 & 4 & 2 \\
4 & 0 & 4 & 3 & 2 & 1 \\
\end{array}
\]

9.6: # 9.56 (a) $[7] + [5] = [7 + 5] = [12] = [1]$
(b) $[7] \cdot [5] = [35] = [2]$
(c) $[-82] + [207] = [6] + [9] = [6 + 9] = [15] = [4]$
(d) $[-82] \cdot [207] = [6] \cdot [9] = [54] = [10]$

9.6: # 9.60 Claim: $n|m$

Proof: Since $a, a + m \in [b] \in \mathbb{Z}_n$, $n|(a + m - a)$ and so $n|m$. 
