# Graph Energy Change Due to Edge Deletion

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**Abstract**

The energy of a graph is the sum of the singular values of its adjacency matrix. We are interested in how the energy of a graph changes when edges are deleted. Examples show that all cases are possible: increased, decreased, unchanged. Our goal is to find possible graph theoretical descriptions and to provide an infinite family of graphs for each case. The main tool is a singular value inequality for complementary submatrices and its equality case.

**Keywords.** Singular value inequality, graph energy.

**AMS subject classifications.** 15A45, 05C50.

## 1 Introduction

Throughout, $G$ will be a simple graph, i.e., a graph with no loop and no multiple edge. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$ respectively. Also let $A(G)$ denote the adjacency matrix of the graph $G$. If $E$ is a subset of $E(G)$, then $G - E$ will denote the subgraph of $G$ with vertex set $V(G)$ but with edge set $E(G) - E$. Such subgraph is also called a *spanning* subgraph of $G$. A subgraph $H$ of $G$ is an *induced* subgraph of $G$ if $H$ contains all edges of $G$ that join two vertices of $H$. Clearly $H$ is induced if and only if $A(H)$ is a principal submatrix of $A(G)$. We write $G - H$ for the graph obtained from $G$ by deleting all vertices of an induced subgraph $H$ and all edges incident with $H$. This is also called the *complement* of $H$ in $G$. Moreover, when no edge of $G$ joins $H$ and its complement $G - H$, we write $G = H \oplus (G - H)$. If $E$ is a set of edges of $G$ such that $G - E$ is the union of two complementary induced subgraphs, then $E$ is called a *cut set* of $G$.

Let $s_j(\cdot)$ denote the singular values of a matrix, and $\lambda_j(\cdot)$ denote the eigenvalues of a matrix. The characteristic polynomial and spectrum of a graph are those of its adjacency

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matrix. The *energy* of a graph $G$ is defined as $\mathcal{E}(G) = \sum_{j=1}^{n} |\lambda_j(A(G))|$ [6]. Since $A(G)$ is a real symmetric matrix, the scalars $|\lambda_j(A(G))|$ are the singular values of $A(G)$ [8]. Hence the energy of a graph is the sum of the singular values of its adjacency matrix. We are interested in comparing the energy of a graph and a subgraph obtained by deleting some of its edges. Examples show that this can increase, decrease, or remain the same (See Example 4.1). In [4], we studied the maximum amount of change. In this paper, we study possible graph theoretical descriptions for different cases. In particular, we address the following questions posed by other authors:

1. *Do there exist graphs such that removing any one edge increases the energy?* [3]

   See Example 4.6 for an infinite family of graphs with this property.

2. *Let $G''$ be a spanning subgraph of a graph $G$. When does the inequality $\mathcal{E}(G'') \leq \mathcal{E}(G)$ hold?* [3]

   See Theorem 3.4 for a sufficient condition.

3. *Characterize the graphs $G$ and their edges $e$ for which $\mathcal{E}(G - \{e\}) \leq \mathcal{E}(G)$.* [5]

   See Theorem 4.2 for a sufficient condition that $\mathcal{E}(G - \{e\}) < \mathcal{E}(G)$.

4. *Which connected graphs have an edge $e$ such that $\mathcal{E}(G - \{e\}) = \mathcal{E}(G)$?* [1]

   See Example 4.8 for an infinite family of graphs with this property.

The rest of the paper is organized as follows. In section 2, we prove a singular value inequality for complementary submatrices and characterize its equality case. Then this inequality is applied in section 3 to obtain results in graph energy change when a cut set is deleted. Section 4 presents several infinite families of graphs, each having an interesting graph energy property when an edge is deleted.

## 2 A singular value inequality

**Lemma 2.1.** Let $C$ be a complex $n \times n$ matrix. Then

$$\left| \sum_{j=1}^{n} \lambda_j(C) \right| \leq \sum_{j=1}^{n} s_j(C).$$

Equality holds if and only if there exists a real scalar $\theta$ such that $e^{\mathbf{i}\theta} C$ is positive semi-definite.

**Proof.** Let $\{z_1, z_2, \ldots, z_n\}$ be an orthonormal set of eigenvectors of $C^*C$ with respect to $s_j^2(C)$, i.e., $C^*Cz_j = s_j^2(C)z_j$. Note that $U = [z_1 \; z_2 \; \cdots \; z_n]$ is a unitary matrix. Now we have

$$\begin{aligned}
\left|\sum_j \lambda_j(C)\right| &= |trC| \\
&= |trU^*CU| \\
&= \left|\sum_j z_j^* C z_j\right| \\
&\le \sum_j |z_j^* C z_j| \qquad \text{by triangle inequality} \\
&\le \sum_j \|Cz_j\| \qquad \text{by Cauchy-Schwarz inequality} \\
&= \sum_j \sqrt{z_j^* C^* C z_j} \\
&= \sum_j s_j(C).
\end{aligned}$$

Equality holds if and only if $|\sum_j z_j^* C z_j| = \sum_j |z_j^* C z_j| = \sum_j \|Cz_j\|$ if and only if there exists $\theta$ such that $z_j^* C z_j = |z_j^* C z_j| e^{-\mathbf{i}\theta}$ and $|z_j^* C z_j| = \|Cz_j\|$ for all $j$ if and only if $z_j^* C z_j = |z_j^* C z_j| e^{-\mathbf{i}\theta}$ and $z_j$'s are eigenvectors of $C$ with respect to eigenvalues $z_j^* C z_j$ if and only if $e^{\mathbf{i}\theta} C$ is positive semi-definite. ■

The inequality in the next theorem is a special case of a more general inequality from [9]. The equality case is new.

**Theorem 2.2.** For a partitioned matrix $C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix}$ where both $A$ and $B$ are square matrices, we have

$$\sum_j s_j(A) + \sum_j s_j(B) \le \sum_j s_j(C).$$

Equality holds if and only if there exist unitary matrices $U$ and $V$ such that $\begin{bmatrix} UA & UX \\ VY & VB \end{bmatrix}$ is positive semi-definite.

**Proof.** By polar decomposition, there exist unitary matrices $U$ and $V$ such that $A' = UA$ and $B' = VB$ are positive semi-definite. Consider the matrix $C' = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} C = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} A & X \\ Y & B \end{bmatrix} = \begin{bmatrix} UA & UX \\ VY & VB \end{bmatrix} = \begin{bmatrix} A' & UX \\ VY & B' \end{bmatrix}$. Now we have

$$\sum_j s_j(A) + \sum_j s_j(B) \quad = \quad \sum_j s_j(A') + \sum_j s_j(B')$$

$$
\begin{aligned}
&= trA' + trB' \qquad \text{because } A' \text{ and } B' \text{ p.s.d.} \\
&= trC' \\
&\leq \sum_j s_j(C') \qquad \text{by Lemma 2.1} \\
&= \sum_j s_j(C),
\end{aligned}
$$

which proves the inequality. Moreover, equality holds if and only if $trC' = \sum_j s_j(C')$ if and only if there exists $\theta$ such that $e^{\mathbf{i}\theta}C'$ is positive semi-definite, by Lemma 2.1. Since $trC'$ is non-negative, $e^{\mathbf{i}\theta} = 1$, i.e., $C'$ is positive semi-definite. Hence equality holds if and only if there exist unitary matrices $U$ and $V$ such that $\begin{bmatrix} UA & UX \\ VY & VB \end{bmatrix}$ is positive semi-definite. ■

**Remark 2.3.** If the matrix $C$ in Theorem 2.2 is real then both unitary matrices $U$ and $V$ can be taken to be real orthogonal in the equality characterization.

**Corollary 2.4.** For a partitioned matrix $C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix}$ where $A$ and $B$ are square matrices, we have

$$\sum_j s_j(A) \leq \sum_j s_j(C).$$

Equality holds if and only if $X, Y$, and $B$ are all zero matrices.

**Proof.** From Theorem 2.2, we have $\sum_j s_j(C) \geq \sum_j s_j(A) + \sum_j s_j(B) \geq \sum_j s_j(A)$. The sufficiency of the equality case is obvious. For the necessity part, we assume that $\sum_j s_j(A) = \sum_j s_j(C)$. It follows that $\sum_j s_j(C) = \sum_j s_j(A) + \sum_j s_j(B) = \sum_j s_j(A)$. Now the second equality implies that $\sum_j s_j(B) = 0$ and so $B = 0$. Moreover the first equality implies, by Theorem 2.2, that there exist unitary matrices $U$ and $V$ such that $\begin{bmatrix} UA & UX \\ VY & VB \end{bmatrix} = \begin{bmatrix} UA & UX \\ VY & 0 \end{bmatrix}$ is positive semi-definite. Consequently, $UX = 0$ and $VY = 0$ [7, section 7.1, exer. 2], so $X = 0$ and $Y = 0$. ■

# 3 Edge set deletion

**Theorem 3.1.** Let $G'$ be an induced subgraph of a simple graph $G$. Then $\mathcal{E}(G') \leq \mathcal{E}(G)$ and equality holds if and only if $E(G') = E(G)$.

**Proof.** Apply Corollary 2.4 to the adjacency matrix $A(G) = \begin{bmatrix} A(G') & X \\ X^T & A(G - G') \end{bmatrix}$. ■

**Corollary 3.2.** For any simple graph $G$ with at least one edge, $\mathcal{E}(G) \geq 2$.

**Proof.** Since $G$ has at least one edge, the complete graph $K_2$ on 2 vertices is an induced subgraph of $G$. Then Theorem 3.1 gives $\mathcal{E}(G) \geq \mathcal{E}(K_2) = 2$. ■

Corollary 3.2 improves a result of Balakrishnan [2, Cor 5.6]: $\mathcal{E}(G) > 1$ for any graph $G$ with at least one edge.

**Example 3.3.** This example shows that the conclusion of Theorem 3.1 may not be true if $G'$ is not an induced subgraph. Let $C_4$ be the cycle graph with 4 vertices. Deleting any edge leaves $P_4$, the path graph with 4 vertices. $P_4$ is not an induced subgraph of $C_4$, and $\mathcal{E}(C_4) = 4 < 2\sqrt{5} = \mathcal{E}(P_4)$.

**Theorem 3.4** If $E$ is a cut set of a simple graph $G$ then $\mathcal{E}(G - E) \leq \mathcal{E}(G)$.

**Proof.** Since $E$ is a cut set of $G$, $G - E = H \oplus K$ where $H$ and $K$ are two complementary induced subgraphs of $G$. Apply Theorem 2.2 to $A(G) = \begin{bmatrix} A(H) & X \\ X^T & A(K) \end{bmatrix}$ to obtain the desired conclusion. ■

It is interesting to characterize the equality case of Theorem 3.4. Using Theorem 2.2, it is equivalent to the existence of orthogonal matrices $U$ and $V$ such that $\begin{bmatrix} UA(H) & UX \\ VX^T & VA(K) \end{bmatrix}$ is positive semi-definite. Unfortunately, this condition does not correspond to any known graph theoretical interpretation. Nonetheless, we give a sufficient (but not necessary) condition, and a necessary (but not sufficient) condition.

**Example 3.5.** For $n \geq 2$, let $G(n,n)$ be a graph consisting of two copies of the complete graph $K_n$ on $n$ vertices with $n$ parallel edges between them. If $E$ is the set of the $n$ parallel edges, then $E$ is a cut set of $G(n,n)$. Note that $\sigma(G(n,n)) = \{n, n-2, 0^{n-1}, (-2)^{n-1}\}$. Hence $\mathcal{E}(G(n,n)) = 4n - 4 = \mathcal{E}(K_n \oplus K_n) = \mathcal{E}(G(n,n) - E)$.

**Theorem 3.6** Let $E$ be the cut set between two complementary induced subgraphs $H$ and $K$ of a graph $G$. Suppose $E$ is not empty and all edges in $E$ are incident to one and only one vertex in $K$, i.e., the edges in $E$ form a star. Then $\mathcal{E}(G - E) < \mathcal{E}(G)$.

**Proof.** Note that $G - E = H \oplus K$ and the edges of $G$ can be ordered so that $A(G) = \begin{bmatrix} A(H) & X \\ X^T & A(K) \end{bmatrix}$ where $A(H)$ is $r \times r$, $A(K)$ is $(n-r) \times (n-r)$, and $X$ is $r \times (n-r)$ with all entries equal to 0 except the first column $x_1$ of $X$ is *nonzero*. By Theorem 3.4, we have $\mathcal{E}(G - E) \leq \mathcal{E}(G)$.

Suppose that $\mathcal{E}(G - E) = \mathcal{E}(G)$. According to the equality case of Theorem 2.2, there exist orthogonal matrices $U$ and $V$ such that

$$\begin{bmatrix} UA(H) & UX \\ VX^T & VA(K) \end{bmatrix} \tag{1}$$

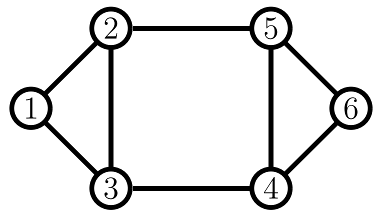


Figure 1: Graph $H$.

is positive semi-definite. From (1), symmetry implies $(UX)^T = VX^T$. Using the special structure of $X$, it follows that $V = \begin{bmatrix} \alpha & 0 \\ 0 & V_1 \end{bmatrix}$ where $V_1$ is orthogonal and $\alpha$ is a scalar with $|\alpha| = 1$. Again from (1), $VA(K)$ is positive semi-definite. Write $A(K) = \begin{bmatrix} 0 & y^T \\ y & K_1 \end{bmatrix}$ and then $VA(K) = \begin{bmatrix} 0 & \alpha y^T \\ V_1 y & V_1 K_1 \end{bmatrix}$. Because the (1,1) entry of $VA(K)$ is zero and $\alpha$ is not zero, as in the proof of Corollary 2.4, we have $\alpha y^T = 0$ and hence $y = 0$. Consequently, $A(K) = \begin{bmatrix} 0 & 0 \\ 0 & K_1 \end{bmatrix}$, hence $A(G) = \begin{bmatrix} A(H) & x_1 & 0 \\ x_1^T & 0 & 0 \\ & & \\ 0 & 0 & K_1 \end{bmatrix}$. Now calculate $\sum_j s_j(A(H)) + \sum_j s_j(K_1) = \sum_j s_j(A(H)) + \sum_j s_j(A(K)) = \mathcal{E}(H) + \mathcal{E}(K) = \mathcal{E}(G - E) = \mathcal{E}(G) = \sum_j s_j \left( \begin{bmatrix} A(H) & x_1 \\ x_1^T & 0 \end{bmatrix} \right) + \sum_j s_j(K_1)$. Hence $\sum_j s_j \left( \begin{bmatrix} A(H) & x_1 \\ x_1^T & 0 \end{bmatrix} \right) = \sum_j s_j(A(H))$. By Corollary 2.4, $x_1 = 0$, hence $E$ is empty, a contradiction. ■

# 4 Single edge deletion

Let $v$ any vertex in a simple graph $G$. Then $G - \{v\}$ is an induced subgraph $G$. Hence, by Theorem 3.1, $\mathcal{E}(G - \{v\}) \leq \mathcal{E}(G)$ and equality holds if and only if $v$ is an isolated vertex. However, the situation is less clear when an edge is deleted. We start with an example showing that the energy of a graph may increase, decrease, or even remain the same when an edge is deleted.

**Example 4.1.** Let $H$ be the graph on 6 vertices in Figure 1. Then

$$\sigma(H) = \{1 + \sqrt{3}, \sqrt{2}, 0, 1 - \sqrt{3}, -\sqrt{2}, -2\}$$

and $\mathcal{E}(H) = 2(1 + \sqrt{2} + \sqrt{3}) \approx 8.2925$. Let $H_1$ be the graph obtained from $H$ by deleting the edge $\{2, 3\}$. Then $\mathcal{E}(H_1) \approx 8.3898 > \mathcal{E}(H)$. Let $H_2$ be the graph obtained from $H$ by deleting the edge $\{1, 2\}$. Then $\mathcal{E}(H_2) \approx 7.7662 < \mathcal{E}(H)$. Let $H_3$ be the graph obtained from $H$ by deleting the edge $\{2,5\}$. Then

$$\sigma(H_3) = \{1+\sqrt{2}, \sqrt{3}, 1-\sqrt{2}, -1, -1, -\sqrt{3}\}$$

and $\mathcal{E}(H_3) = 2(1+\sqrt{2}+\sqrt{3}) = \mathcal{E}(H)$.

The following theorem gives a sufficient condition for the graph energy to decrease when an edge is deleted. A singleton cut set is called a *bridge*.

**Theorem 4.2.** If $\{e\}$ is a bridge in a simple graph $G$, then $\mathcal{E}(G-\{e\}) < \mathcal{E}(G)$.

**Proof.** Take $E = \{e\}$ in Theorem 3.6. ■

**Corollary 4.3.** Let $e$ be an edge of a tree $T$. Then $\mathcal{E}(T-\{e\}) < \mathcal{E}(T)$.

**Remark 4.4.** Let $P_n$ denote the path graph on $n$ vertices. Applying Corollary 4.3, we obtain the inequality

$$\mathcal{E}(P_n) > \mathcal{E}(P_k) + \mathcal{E}(P_{n-k})$$

for $1 \le k \le n-1$. Since the spectrum of $P_n$ is $\sigma(P_n) = \left\{2\cos(\frac{j\pi}{n+1}) : 1 \le j \le n\right\}$, the above inequality becomes

$$\sum_{j=1}^{n} \left|\cos\left(\frac{j\pi}{n+1}\right)\right| > \sum_{j=1}^{k} \left|\cos\left(\frac{j\pi}{k+1}\right)\right| + \sum_{j=1}^{n-k} \left|\cos\left(\frac{j\pi}{n-k+1}\right)\right|$$

for $1 \le k \le n-1$. This may not be easy to prove directly.

Finally we include three infinite families of graphs, each having an interesting property with respect to graph energy change.

**Example 4.5.** Here is an infinite family with the property that deleting *any* edge will decrease the energy. Let $K_n$ be the complete graph on $n$ vertices. Then $\sigma(K_n) = \{(-1)^{(n-1)}, n-1\}$, so $\mathcal{E}(K_n) = 2n-2$. Let $e$ be any edge in $K_n$. For $n \ge 3$, the characteristic polynomial of $K_n - \{e\}$ is $x(x+1)^{n-3}[x^2-(n-3)x-(2n-4)]$ and so $\sigma(K_n-\{e\}) = \{\frac{n-3\pm\sqrt{n^2+2n-7}}{2}, 0, (-1)^{(n-3)}\}$. It follows that $\mathcal{E}(K_n-\{e\}) = n-3+\sqrt{n^2+2n-7}$. Thus $\mathcal{E}(K_n) > \mathcal{E}(K_n-\{e\})$.

**Example 4.6** Here is an infinite family with the property that deleting *any* edge will increase the energy. Let $K_{n,n}$ be the regular complete bipartite graph on $2n$ vertices. Then $\sigma(K_{n,n}) = \{n, 0^{(2n-2)}, -n\}$ and so $\mathcal{E}(K_{n,n}) = 2n$. Choose any edge $e$ and order the vertices of $K_{n,n}$ so that the edge $e = \{1, n+1\}$. For $n \ge 2$, let $A$ be the $n \times n$ matrix with all entries equal to 1 except the $(1,1)$ entry is 0. Then $A(K_{n,n}-\{e\}) = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$. Since $\operatorname{rank}(A) = 2$, $A$ has eigenvalue 0 with multiplicity $n-2$. On the other hand, $A^3-(n-1)A^2-(n-1)A = 0$, so $A$ must have minimal polynomial $x^3-(n-1)x^2-(n-1)x = 0$ for $n \ge 3$ and $x^2-x-1$ for $n=2$. Consequently, $\sigma(A) = \{\frac{n-1\pm\sqrt{n^2+2n-3}}{2}, 0^{(n-2)}\}$ and $\sigma(K_{n,n}-\{e\}) = \pm\{\frac{\sqrt{n^2+2n-3}\pm(n-1)}{2}, 0^{(n-2)}\}$. It follows that $\mathcal{E}(K_{n,n}-\{e\}) = 2\sqrt{n^2+2n-3}$. Thus $\mathcal{E}(K_{n,n}-\{e\}) > \mathcal{E}(K_{n,n})$.

**Lemma 4.7.** For $n \geq 2$, let $D$ be a diagonal matrix with diagonal entries $d_1, \ldots, d_n$ and $J$ be the matrix with all entries equal to 1. Then $\det(D+J) = d_1 \cdots d_n + p_1 + \cdots + p_n$ where $p_j$ denotes the product of all $d_1, \ldots, d_n$ except $d_j$.

**Proof.** *Case 1:* When there are more than one $d_i$ equal to zero, then $d_1 \cdots d_n + p_1 + \cdots + p_n = 0$. Note that $D+J$ will have two identical rows, and so $\det(D+J) = 0$.

*Case 2:* When there is exactly one $d_i$ equal to zero, say $d_n = 0$, then $d_1 \cdots d_n + p_1 + \cdots + p_n = d_1, \cdots, d_{n-1}$. Now subtract the $n$-th row from all other rows in $D+J$, we have a lower triangular matrix whose diagonal entries are $d_1, \ldots, d_{n-1}, 1$. Hence $\det(D+J) = d_1, \ldots, d_{n-1}$.

*Case 3:* When there is NO $d_i$ equal to zero, then $D^{-1}$ is also a diagonal matrix with diagonal entries $d_1^{-1}, \ldots, d_n^{-1}$. Since $D^{-1}J$ has rank one, its spectrum $\sigma(D^{-1}J) = \{\frac{1}{d_1} + \cdots + \frac{1}{d_n}, 0^{(n-1)}\}$, so $\sigma(I + D^{-1}J) = \{1 + \frac{1}{d_1} + \cdots + \frac{1}{d_n}, 1^{(n-1)}\}$. Hence

$$\begin{aligned}\det(D+J) &= \det(D)\det(I + D^{-1}J) \\ &= (d_1 \cdots d_n)(1 + \frac{1}{d_1} + \cdots + \frac{1}{d_n}) \\ &= d_1 \cdots d_n + p_1 + \cdots + p_n \quad \blacksquare\end{aligned}$$

**Example 4.8.** Here is an infinite family with the property that deleting a *certain* edge does not change the energy. For $n \geq 2$, let $G(n,r)$ be the graph with two disjoint copies of the complete graph $K_n$ joined by $0 \leq r \leq n$ parallel edges. Then $\mathcal{E}(G(n,r)) = \mathcal{E}(G(n, n-r))$ for all $r$. In particular, when $n = 2k+1$ and $r = k+1$ for $k \geq 1$, removing any one of the $r$ parallel edges in the cut set does not change the energy.

**Proof.** The case $r = 0$ and $r = n$ has been done in Example 3.5. For the rest of this proof, we assume that $1 \leq r \leq n-1$. The adjacency matrix of $G(n,r)$ is

$$A(G(n,r)) = \begin{bmatrix} J-I & R \\ R & J-I \end{bmatrix}$$

where $J$ is the all-one matrix, $I$ is the identity matrix and $R$ is the diagonal matrix with $r$ 1's and $n-r$ 0's on its diagonal. Hence the characteristic polynomial of $G(n,r)$ is

$$\begin{aligned} &\det \begin{bmatrix} J-I-xI & R \\ R & J-I-xI \end{bmatrix} \\ =\ &\det(J-I-xI-R)\det(J-I-xI+R) \\ =\ &\det(D_1+J)\det(D_2+J) \end{aligned}$$

where both $D_1 = -R-(1+x)I$ and $D_2 = R-(1+x)I$ are diagonal matrices. Using Lemma 4.7 twice, we compute the characteristic polynomial of $G(n,r)$ as

$$x^{r-1}(x+2)^{r-1}(x+1)^{2n-2r-2}[x^2-(n-1)x-r][x^2-(n-3)x-(2n-r-2)],$$

so $\sigma(G(n,r))$ is

$$\left\{\frac{n-1\pm\sqrt{n^2+1+2(2r-n)}}{2}, \frac{n-3\pm\sqrt{n^2+1+2(n-2r)}}{2}, 0^{r-1}, (-1)^{2n-2r-2}, (-2)^{r-1}\right\}.$$

Therefore the energy of $G(n,r)$ is

$$\mathcal{E}(G(n,r)) = 2n-4+\sqrt{n^2+1+2(2r-n)}+\sqrt{n^2+1+2(n-2r)}.$$

It follows easily that $\mathcal{E}(G(n,r)) = \mathcal{E}(G(n,n-r))$. In particular, $\mathcal{E}(G(2k+1,k+1)) = \mathcal{E}(G(2k+1,k))$ for any integer $k \geq 1$. Consequently, if $G = G(2k+1,k+1)$ and $e$ is one of the parallel edges then $G-\{e\} = G(2k+1,k)$. Thus $\mathcal{E}(G) = \mathcal{E}(G-\{e\})$. ■

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