

Graph Energy Change Due to Edge Deletion

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Abstract

The energy of a graph is the sum of the singular values of its adjacency matrix. We are interested in how the energy of a graph changes when edges are deleted. Examples show that all cases are possible: increased, decreased, unchanged. Our goal is to find possible graph theoretical descriptions and to provide an infinite family of graphs for each case. The main tool is a singular value inequality for complementary submatrices and its equality case.

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1 Introduction

Throughout, G will be a simple graph, i.e., a graph with no loop and no multiple edge. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of G respectively. Also let $A(G)$ denote the adjacency matrix of the graph G . If E is a subset of $E(G)$, then $G - E$ will denote the subgraph of G with vertex set $V(G)$ but with edge set $E(G) - E$. Such subgraph is also called a *spanning* subgraph of G . A subgraph H of G is an *induced* subgraph of G if H contains all edges of G that join two vertices of H . Clearly H is induced if and only if $A(H)$ is a principal submatrix of $A(G)$. We write $G - H$ for the graph obtained from G by deleting all vertices of an induced subgraph H and all edges incident with H . This is also called the *complement* of H in G . Moreover, when no edge of G joins H and its complement $G - H$, we write $G = H \oplus (G - H)$. If E is a set of edges of G such that $G - E$ is the union of two complementary induced subgraphs, then E is called a *cut set* of G .

Let $s_j(\cdot)$ denote the singular values of a matrix, and $\lambda_j(\cdot)$ denote the eigenvalues of a matrix. The characteristic polynomial and spectrum of a graph are those of its adjacency

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matrix. The *energy* of a graph G is defined as $\mathcal{E}(G) = \sum_{j=1}^n |\lambda_j(A(G))|$ [6]. Since $A(G)$ is a real symmetric matrix, the scalars $|\lambda_j(A(G))|$ are the singular values of $A(G)$ [8]. Hence the energy of a graph is the sum of the singular values of its adjacency matrix. We are interested in comparing the energy of a graph and a subgraph obtained by deleting some of its edges. Examples show that this can increase, decrease, or remain the same (See Example 4.1). In [4], we studied the maximum amount of change. In this paper, we study possible graph theoretical descriptions for different cases. In particular, we address the following questions posed by other authors:

1. *Do there exist graphs such that removing any one edge increases the energy?* [3]

See Example 4.6 for an infinite family of graphs with this property.

2. *Let G'' be a spanning subgraph of a graph G . When does the inequality $\mathcal{E}(G'') \leq \mathcal{E}(G)$ hold?* [3]

See Theorem 3.4 for a sufficient condition.

3. *Characterize the graphs G and their edges e for which $\mathcal{E}(G - \{e\}) \leq \mathcal{E}(G)$.* [5]

See Theorem 4.2 for a sufficient condition that $\mathcal{E}(G - \{e\}) < \mathcal{E}(G)$.

4. *Which connected graphs have an edge e such that $\mathcal{E}(G - \{e\}) = \mathcal{E}(G)$?* [1]

See Example 4.8 for an infinite family of graphs with this property.

The rest of the paper is organized as follows. In section 2, we prove a singular value inequality for complementary submatrices and characterize its equality case. Then this inequality is applied in section 3 to obtain results in graph energy change when a cut set is deleted. Section 4 presents several infinite families of graphs, each having an interesting graph energy property when an edge is deleted.

2 A singular value inequality

Lemma 2.1. Let C be a complex $n \times n$ matrix. Then

$$\left| \sum_{j=1}^n \lambda_j(C) \right| \leq \sum_{j=1}^n s_j(C).$$

Equality holds if and only if there exists a real scalar θ such that $e^{i\theta}C$ is positive semi-definite.

Proof. Let $\{z_1, z_2, \dots, z_n\}$ be an orthonormal set of eigenvectors of C^*C with respect to $s_j^2(C)$, i.e., $C^*Cz_j = s_j^2(C)z_j$. Note that $U = [z_1 \ z_2 \ \dots \ z_n]$ is a unitary matrix. Now

we have

$$\begin{aligned}
\left| \sum_j \lambda_j(C) \right| &= |\operatorname{tr} C| \\
&= |\operatorname{tr} U^* C U| \\
&= \left| \sum_j z_j^* C z_j \right| \\
&\leq \sum_j |z_j^* C z_j| \quad \text{by triangle inequality} \\
&\leq \sum_j \|C z_j\| \quad \text{by Cauchy-Schwarz inequality} \\
&= \sum_j \sqrt{z_j^* C^* C z_j} \\
&= \sum_j s_j(C).
\end{aligned}$$

Equality holds if and only if $|\sum_j z_j^* C z_j| = \sum_j |z_j^* C z_j| = \sum_j \|C z_j\|$ if and only if there exists θ such that $z_j^* C z_j = |z_j^* C z_j| e^{-i\theta}$ and $|z_j^* C z_j| = \|C z_j\|$ for all j if and only if $z_j^* C z_j = |z_j^* C z_j| e^{-i\theta}$ and z_j 's are eigenvectors of C with respect to eigenvalues $z_j^* C z_j$ if and only if $e^{i\theta} C$ is positive semi-definite. ■

The inequality in the next theorem is a special case of a more general inequality from [9]. The equality case is new.

Theorem 2.2. For a partitioned matrix $C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix}$ where both A and B are square matrices, we have

$$\sum_j s_j(A) + \sum_j s_j(B) \leq \sum_j s_j(C).$$

Equality holds if and only if there exist unitary matrices U and V such that $\begin{bmatrix} UA & UX \\ VY & VB \end{bmatrix}$ is positive semi-definite.

Proof. By polar decomposition, there exist unitary matrices U and V such that $A' = UA$ and $B' = VB$ are positive semi-definite. Consider the matrix $C' = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} C = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} A & X \\ Y & B \end{bmatrix} = \begin{bmatrix} UA & UX \\ VY & VB \end{bmatrix} = \begin{bmatrix} A' & UX \\ VY & B' \end{bmatrix}$. Now we have

$$\sum_j s_j(A) + \sum_j s_j(B) = \sum_j s_j(A') + \sum_j s_j(B')$$

$$\begin{aligned}
&= \operatorname{tr}A' + \operatorname{tr}B' \quad \text{because } A' \text{ and } B' \text{ p.s.d.} \\
&= \operatorname{tr}C' \\
&\leq \sum_j s_j(C') \quad \text{by Lemma 2.1} \\
&= \sum_j s_j(C),
\end{aligned}$$

which proves the inequality. Moreover, equality holds if and only if $\operatorname{tr}C' = \sum_j s_j(C')$ if and only if there exists θ such that $e^{i\theta}C'$ is positive semi-definite, by Lemma 2.1. Since $\operatorname{tr}C'$ is non-negative, $e^{i\theta} = 1$, i.e., C' is positive semi-definite. Hence equality holds if and only if there exist unitary matrices U and V such that $\begin{bmatrix} UA & UX \\ VY & VB \end{bmatrix}$ is positive semi-definite. ■

Remark 2.3. If the matrix C in Theorem 2.2 is real then both unitary matrices U and V can be taken to be real orthogonal in the equality characterization.

Corollary 2.4. For a partitioned matrix $C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix}$ where A and B are square matrices, we have

$$\sum_j s_j(A) \leq \sum_j s_j(C).$$

Equality holds if and only if X, Y , and B are all zero matrices.

Proof. From Theorem 2.2, we have $\sum_j s_j(C) \geq \sum_j s_j(A) + \sum_j s_j(B) \geq \sum_j s_j(A)$. The sufficiency of the equality case is obvious. For the necessity part, we assume that $\sum_j s_j(A) = \sum_j s_j(C)$. It follows that $\sum_j s_j(C) = \sum_j s_j(A) + \sum_j s_j(B) = \sum_j s_j(A)$. Now the second equality implies that $\sum_j s_j(B) = 0$ and so $B = 0$. Moreover the first equality implies, by Theorem 2.2, that there exist unitary matrices U and V such that $\begin{bmatrix} UA & UX \\ VY & VB \end{bmatrix} = \begin{bmatrix} UA & UX \\ VY & 0 \end{bmatrix}$ is positive semi-definite. Consequently, $UX = 0$ and $VY = 0$ [7, section 7.1, exer. 2], so $X = 0$ and $Y = 0$. ■

3 Edge set deletion

Theorem 3.1. Let G' be an induced subgraph of a simple graph G . Then $\mathcal{E}(G') \leq \mathcal{E}(G)$ and equality holds if and only if $E(G') = E(G)$.

Proof. Apply Corollary 2.4 to the adjacency matrix $A(G) = \begin{bmatrix} A(G') & X \\ X^T & A(G - G') \end{bmatrix}$. ■

Corollary 3.2. For any simple graph G with at least one edge, $\mathcal{E}(G) \geq 2$.

Proof. Since G has at least one edge, the complete graph K_2 on 2 vertices is an induced subgraph of G . Then Theorem 3.1 gives $\mathcal{E}(G) \geq \mathcal{E}(K_2) = 2$. ■

Corollary 3.2 improves a result of Balakrishnan [2, Cor 5.6]: $\mathcal{E}(G) > 1$ for any graph G with at least one edge.

Example 3.3. This example shows that the conclusion of Theorem 3.1 may not be true if G' is not an induced subgraph. Let C_4 be the cycle graph with 4 vertices. Deleting any edge leaves P_4 , the path graph with 4 vertices. P_4 is not an induced subgraph of C_4 , and $\mathcal{E}(C_4) = 4 < 2\sqrt{5} = \mathcal{E}(P_4)$.

Theorem 3.4 If E is a cut set of a simple graph G then $\mathcal{E}(G - E) \leq \mathcal{E}(G)$.

Proof. Since E is a cut set of G , $G - E = H \oplus K$ where H and K are two complementary induced subgraphs of G . Apply Theorem 2.2 to $A(G) = \begin{bmatrix} A(H) & X \\ X^T & A(K) \end{bmatrix}$ to obtain the desired conclusion. ■

It is interesting to characterize the equality case of Theorem 3.4. Using Theorem 2.2, it is equivalent to the existence of orthogonal matrices U and V such that $\begin{bmatrix} UA(H) & UX \\ VX^T & VA(K) \end{bmatrix}$ is positive semi-definite. Unfortunately, this condition does not correspond to any known graph theoretical interpretation. Nonetheless, we give a sufficient (but not necessary) condition, and a necessary (but not sufficient) condition.

Example 3.5. For $n \geq 2$, let $G(n, n)$ be a graph consisting of two copies of the complete graph K_n on n vertices with n parallel edges between them. If E is the set of the n parallel edges, then E is a cut set of $G(n, n)$. Note that $\sigma(G(n, n)) = \{n, n - 2, 0^{n-1}, (-2)^{n-1}\}$. Hence $\mathcal{E}(G(n, n)) = 4n - 4 = \mathcal{E}(K_n \oplus K_n) = \mathcal{E}(G(n, n) - E)$.

Theorem 3.6 Let E be the cut set between two complementary induced subgraphs H and K of a graph G . Suppose E is not empty and all edges in E are incident to one and only one vertex in K , i.e., the edges in E form a star. Then $\mathcal{E}(G - E) < \mathcal{E}(G)$.

Proof. Note that $G - E = H \oplus K$ and the edges of G can be ordered so that $A(G) = \begin{bmatrix} A(H) & X \\ X^T & A(K) \end{bmatrix}$ where $A(H)$ is $r \times r$, $A(K)$ is $(n - r) \times (n - r)$, and X is $r \times (n - r)$ with all entries equal to 0 except the first column x_1 of X is *nonzero*. By Theorem 3.4, we have $\mathcal{E}(G - E) \leq \mathcal{E}(G)$.

Suppose that $\mathcal{E}(G - E) = \mathcal{E}(G)$. According to the equality case of Theorem 2.2, there exist orthogonal matrices U and V such that

$$\begin{bmatrix} UA(H) & UX \\ VX^T & VA(K) \end{bmatrix} \tag{1}$$

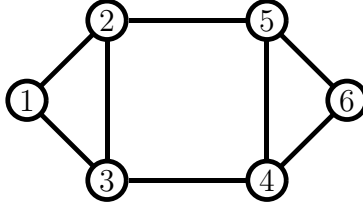


Figure 1: Graph H .

is positive semi-definite. From (1), symmetry implies $(UX)^T = VX^T$. Using the special structure of X , it follows that $V = \begin{bmatrix} \alpha & 0 \\ 0 & V_1 \end{bmatrix}$ where V_1 is orthogonal and α is a scalar with $|\alpha| = 1$. Again from (1), $VA(K)$ is positive semi-definite. Write $A(K) = \begin{bmatrix} 0 & y^T \\ y & K_1 \end{bmatrix}$ and then $VA(K) = \begin{bmatrix} 0 & \alpha y^T \\ V_1 y & V_1 K_1 \end{bmatrix}$. Because the (1,1) entry of $VA(K)$ is zero and α is not zero, as in the proof of Corollary 2.4, we have $\alpha y^T = 0$ and hence $y = 0$.

Consequently, $A(K) = \begin{bmatrix} 0 & 0 \\ 0 & K_1 \end{bmatrix}$, hence $A(G) = \begin{bmatrix} A(H) & x_1 & 0 \\ x_1^T & 0 & 0 \\ 0 & 0 & K_1 \end{bmatrix}$. Now calculate $\sum_j s_j(A(H)) + \sum_j s_j(K_1) = \sum_j s_j(A(H)) + \sum_j s_j(A(K)) = \mathcal{E}(H) + \mathcal{E}(K) = \mathcal{E}(G - E) = \mathcal{E}(G) = \sum_j s_j \left(\begin{bmatrix} A(H) & x_1 \\ x_1^T & 0 \end{bmatrix} \right) + \sum_j s_j(K_1)$. Hence $\sum_j s_j \left(\begin{bmatrix} A(H) & x_1 \\ x_1^T & 0 \end{bmatrix} \right) = \sum_j s_j(A(H))$. By Corollary 2.4, $x_1 = 0$, hence E is empty, a contradiction. ■

4 Single edge deletion

Let v any vertex in a simple graph G . Then $G - \{v\}$ is an induced subgraph G . Hence, by Theorem 3.1, $\mathcal{E}(G - \{v\}) \leq \mathcal{E}(G)$ and equality holds if and only if v is an isolated vertex. However, the situation is less clear when an edge is deleted. We start with an example showing that the energy of a graph may increase, decrease, or even remain the same when an edge is deleted.

Example 4.1. Let H be the graph on 6 vertices in Figure 1. Then

$$\sigma(H) = \{1 + \sqrt{3}, \sqrt{2}, 0, 1 - \sqrt{3}, -\sqrt{2}, -2\}$$

and $\mathcal{E}(H) = 2(1 + \sqrt{2} + \sqrt{3}) \approx 8.2925$. Let H_1 be the graph obtained from H by deleting the edge $\{2, 3\}$. Then $\mathcal{E}(H_1) \approx 8.3898 > \mathcal{E}(H)$. Let H_2 be the graph obtained from H by deleting the edge $\{1, 2\}$. Then $\mathcal{E}(H_2) \approx 7.7662 < \mathcal{E}(H)$. Let H_3 be the

graph obtained from H by deleting the edge $\{2, 5\}$. Then

$$\sigma(H_3) = \{1 + \sqrt{2}, \sqrt{3}, 1 - \sqrt{2}, -1, -1, -\sqrt{3}\}$$

and $\mathcal{E}(H_3) = 2(1 + \sqrt{2} + \sqrt{3}) = \mathcal{E}(H)$.

The following theorem gives a sufficient condition for the graph energy to decrease when an edge is deleted. A singleton cut set is called a *bridge*.

Theorem 4.2. If $\{e\}$ is a bridge in a simple graph G , then $\mathcal{E}(G - \{e\}) < \mathcal{E}(G)$.

Proof. Take $E = \{e\}$ in Theorem 3.6. ■

Corollary 4.3. Let e be an edge of a tree T . Then $\mathcal{E}(T - \{e\}) < \mathcal{E}(T)$.

Remark 4.4. Let P_n denote the path graph on n vertices. Applying Corollary 4.3, we obtain the inequality

$$\mathcal{E}(P_n) > \mathcal{E}(P_k) + \mathcal{E}(P_{n-k})$$

for $1 \leq k \leq n - 1$. Since the spectrum of P_n is $\sigma(P_n) = \{2 \cos(\frac{j\pi}{n+1}) : 1 \leq j \leq n\}$, the above inequality becomes

$$\sum_{j=1}^n \left| \cos\left(\frac{j\pi}{n+1}\right) \right| > \sum_{j=1}^k \left| \cos\left(\frac{j\pi}{k+1}\right) \right| + \sum_{j=1}^{n-k} \left| \cos\left(\frac{j\pi}{n-k+1}\right) \right|$$

for $1 \leq k \leq n - 1$. This may not be easy to prove directly.

Finally we include three infinite families of graphs, each having an interesting property with respect to graph energy change.

Example 4.5. Here is an infinite family with the property that deleting *any* edge will decrease the energy. Let K_n be the complete graph on n vertices. Then $\sigma(K_n) = \{(-1)^{(n-1)}, n - 1\}$, so $\mathcal{E}(K_n) = 2n - 2$. Let e be any edge in K_n . For $n \geq 3$, the characteristic polynomial of $K_n - \{e\}$ is $x(x+1)^{n-3}[x^2 - (n-3)x - (2n-4)]$ and so $\sigma(K_n - \{e\}) = \{\frac{n-3 \pm \sqrt{n^2+2n-7}}{2}, 0, (-1)^{(n-3)}\}$. It follows that $\mathcal{E}(K_n - \{e\}) = n - 3 + \sqrt{n^2 + 2n - 7}$. Thus $\mathcal{E}(K_n) > \mathcal{E}(K_n - \{e\})$.

Example 4.6 Here is an infinite family with the property that deleting *any* edge will increase the energy. Let $K_{n,n}$ be the regular complete bipartite graph on $2n$ vertices. Then $\sigma(K_{n,n}) = \{n, 0^{(2n-2)}, -n\}$ and so $\mathcal{E}(K_{n,n}) = 2n$. Choose any edge e and order the vertices of $K_{n,n}$ so that the edge $e = \{1, n+1\}$. For $n \geq 2$, let A be the $n \times n$ matrix with all entries equal to 1 except the $(1, 1)$ entry is 0. Then $A(K_{n,n} - \{e\}) = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$. Since $\text{rank}(A) = 2$, A has eigenvalue 0 with multiplicity $n - 2$. On the other hand, $A^3 - (n-1)A^2 - (n-1)A = 0$, so A must have minimal polynomial $x^3 - (n-1)x^2 - (n-1)x = 0$ for $n \geq 3$ and $x^2 - x - 1$ for $n = 2$. Consequently, $\sigma(A) = \{\frac{n-1 \pm \sqrt{n^2+2n-3}}{2}, 0^{(n-2)}\}$ and $\sigma(K_{n,n} - \{e\}) = \pm \{\frac{\sqrt{n^2+2n-3} \pm (n-1)}{2}, 0^{(n-2)}\}$. It follows that $\mathcal{E}(K_{n,n} - \{e\}) = 2\sqrt{n^2 + 2n - 3}$. Thus $\mathcal{E}(K_{n,n} - \{e\}) > \mathcal{E}(K_{n,n})$.

Lemma 4.7. For $n \geq 2$, let D be a diagonal matrix with diagonal entries d_1, \dots, d_n and J be the matrix with all entries equal to 1. Then $\det(D + J) = d_1 \cdots d_n + p_1 + \cdots + p_n$ where p_j denotes the product of all d_1, \dots, d_n except d_j .

Proof. *Case 1:* When there are more than one d_i equal to zero, then $d_1 \cdots d_n + p_1 + \cdots + p_n = 0$. Note that $D + J$ will have two identical rows, and so $\det(D + J) = 0$.

Case 2: When there is exactly one d_i equal to zero, say $d_n = 0$, then $d_1 \cdots d_n + p_1 + \cdots + p_n = d_1 \cdots d_{n-1}$. Now subtract the n -th row from all other rows in $D + J$, we have a lower triangular matrix whose diagonal entries are $d_1, \dots, d_{n-1}, 1$. Hence $\det(D + J) = d_1 \cdots d_{n-1}$.

Case 3: When there is NO d_i equal to zero, then D^{-1} is also a diagonal matrix with diagonal entries $d_1^{-1}, \dots, d_n^{-1}$. Since $D^{-1}J$ has rank one, its spectrum $\sigma(D^{-1}J) = \{\frac{1}{d_1} + \cdots + \frac{1}{d_n}, 0^{(n-1)}\}$, so $\sigma(I + D^{-1}J) = \{1 + \frac{1}{d_1} + \cdots + \frac{1}{d_n}, 1^{(n-1)}\}$. Hence

$$\begin{aligned} \det(D + J) &= \det(D) \det(I + D^{-1}J) \\ &= (d_1 \cdots d_n) \left(1 + \frac{1}{d_1} + \cdots + \frac{1}{d_n}\right) \\ &= d_1 \cdots d_n + p_1 + \cdots + p_n \quad \blacksquare \end{aligned}$$

Example 4.8. Here is an infinite family with the property that deleting a *certain* edge does not change the energy. For $n \geq 2$, let $G(n, r)$ be the graph with two disjoint copies of the complete graph K_n joined by $0 \leq r \leq n$ parallel edges. Then $\mathcal{E}(G(n, r)) = \mathcal{E}(G(n, n - r))$ for all r . In particular, when $n = 2k + 1$ and $r = k + 1$ for $k \geq 1$, removing any one of the r parallel edges in the cut set does not change the energy.

Proof. The case $r = 0$ and $r = n$ has been done in Example 3.5. For the rest of this proof, we assume that $1 \leq r \leq n - 1$. The adjacency matrix of $G(n, r)$ is

$$A(G(n, r)) = \begin{bmatrix} J - I & R \\ R & J - I \end{bmatrix}$$

where J is the all-one matrix, I is the identity matrix and R is the diagonal matrix with r 1's and $n - r$ 0's on its diagonal. Hence the characteristic polynomial of $G(n, r)$ is

$$\begin{aligned} &\det \begin{bmatrix} J - I - xI & R \\ R & J - I - xI \end{bmatrix} \\ &= \det(J - I - xI - R) \det(J - I - xI + R) \\ &= \det(D_1 + J) \det(D_2 + J) \end{aligned}$$

where both $D_1 = -R - (1 + x)I$ and $D_2 = R - (1 + x)I$ are diagonal matrices. Using Lemma 4.7 twice, we compute the characteristic polynomial of $G(n, r)$ as

$$x^{r-1}(x+2)^{r-1}(x+1)^{2n-2r-2}[x^2 - (n-1)x - r][x^2 - (n-3)x - (2n-r-2)],$$

so $\sigma(G(n, r))$ is

$$\left\{ \frac{n-1 \pm \sqrt{n^2+1+2(2r-n)}}{2}, \frac{n-3 \pm \sqrt{n^2+1+2(n-2r)}}{2}, 0^{r-1}, (-1)^{2n-2r-2}, (-2)^{r-1} \right\}.$$

Therefore the energy of $G(n, r)$ is

$$\mathcal{E}(G(n, r)) = 2n - 4 + \sqrt{n^2 + 1 + 2(2r - n)} + \sqrt{n^2 + 1 + 2(n - 2r)}.$$

It follows easily that $\mathcal{E}(G(n, r)) = \mathcal{E}(G(n, n - r))$. In particular, $\mathcal{E}(G(2k + 1, k + 1)) = \mathcal{E}(G(2k + 1, k))$ for any integer $k \geq 1$. Consequently, if $G = G(2k + 1, k + 1)$ and e is one of the parallel edges then $G - \{e\} = G(2k + 1, k)$. Thus $\mathcal{E}(G) = \mathcal{E}(G - \{e\})$. ■

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