Lecture 10

Part I

Convergence domains for the
Campbell Baker Hausdorff formula

Let $x$ and $y$ be noncommutative indeterminates, and consider
the element $z$ for which $e^x e^y = e^z$. We already know that there is
such an element. And we learned in Lecture 1 about various ways of
presenting it. The question we ask here is this: Do the various
infinite series presentations of $z$ converge if the symbols $x$ and $y$ are
replaced by matrices?

Now

$$z = \log(1 + w) = w - \frac{1}{2} w^2 + \frac{1}{3} w^3 + ..., $$

where

$$w = w(x,y) = e^x e^y - 1.$$ 

Since the exponential series is globally convergent, $w(A,B)$ exists for
any matrices $A$ and $B$. But the log series is not globally convergent,
the unit disk around the origin is the domain of convergence, so we
expect $\log(1 + w)$ to be convergent only when each eigenvalue of $w$ is
in the circle of radius 1 centered at 0. This already proves that
$z(A,B)$ exists for $A$ and $B$ near zero, but does not prove that it is a Lie
element, a deeper result, and does not estimate the size of the
neighbourhood of zero within which $z(A,B)$ will exist. We wish to
address the latter point. (A Lie element is one expressible in Lie
products $[u,v] = uv-vu$.)

There are various estimates in the literature for the
convergence domain of $z = z(A,B)$. It is known, for example, that the
series is not globally convergent. Here is a standard estimate, from
Bourbaki, *Lie groups and algebras*. Let $\| \|$ be a norm on matrices
compatible with Lie multiplication, that is

$$\|[A,B]\| \leq \|A\|\|B\|$$

for all matrices $A$, $B$, where $[A,B] = AB-BA$ is the Lie product. Note
that a norm compatible with Lie multiplication is easily obtained...
from a norm compatible with associative multiplication, in the following way: If

$$\|AB\|_a \leq \|A\|_a \|B\|_a,$$

is a norm compatible with associative multiplication, then

$$2 \|\{A,B\}\|_a \leq 2 \|A\|_a \cdot 2 \|B\|_a,$$

so that a norm compatible with Lie multiplication is

$$\|A\|_{\text{Lie}} = 2 \|A\|_a.$$

We shall say that this Lie norm is derived from an associative norm. Conceivably there are Lie norms (norms for which $\|\{A,B\}\| \leq \|A\| \|B\|$) not derived from an associative norm.

Now Bourbaki's estimate is that convergence of the Dynkin series for $z(A,B)$ occurs for matrices $A$, $B$ if there is a Lie norm for which

$$\|A\|_{\text{Lie}} \leq \frac{1}{2} \log 2 = .34657\ldots.$$  

(and the same bound for $\|B\|_{\text{Lie}}$.) We wish to investigate this matter, taking advantage of other presentations of $z$. First, however, let me show how the bound just given is derived.

By a Taylor series, it is easy to show, for a norm compatible with associative multiplication, that

$$\|w(A,B)\| = \|e^A e^B - 1\| \leq e^{\|A\|} e^{\|B\|} - 1 = e^{\|A\| + \|B\|} - 1.$$  

Therefore, the majorizing series obtained from $z(A,B)$ by applying an associative norm to the individual terms will converge if $\exp(\|A\| + \|B\|) - 1 < 1$, yielding the estimate that $\|A\|_A$ and $\|B\|_A$, each $< \frac{1}{2} \log 2$ will suffice for convergence.

Now, a not obvious but true fact is that $z$ is a Lie series, and moreover it is a Lie series obtained in this way: Replace each associative word (in the letters $x$ and $y$) with the corresponding left standardized word in Lie brackets using the same letters, and divide by the word length. For example, replace the associative word
with

\[ \frac{1}{6} [[[[[[x,y],y],y],y],x],y]. \]  \hspace{1cm} (2)

The series so obtained is a correct presentation of the exponent \( z \) in Lie products. Now, let us use a Lie norm to majorize any Lie word in the Lie presentation of \( z(A,B) \), for example, to majorize (2), in just the same way as an associative norm majorizes an associative word such as (1). The majorization of (1) by an associative norm produces

\[ \|A\|^2 \|B\|^4, \]

whereas the majorization of (2) under a Lie norm yields

\[ \frac{1}{6} \|A\|^2 \|B\|^4. \]

This means the majorizations of the associative version of \( z(A,B) \) under an associative norm and the Lie version of \( z(A,B) \) under a Lie norm yield the same majorizing series, apart or a factor \( 1/n \) on the degree \( n \) terms obtained from majorization of the Lie series. Now a factor \( 1/n \) on degree \( n \) terms does not affect convergence, and therefore the Lie presentation of \( z(A,B) \) converges absolutely under any Lie norm for which

\[ \|A\|_{Lie} < \frac{1}{2} \log 2, \|B\|_{Lie} < \frac{1}{2} \log 2. \]

With some cleaning up of details, this is a correct proof.

Our starting point will be the Goldberg series,

\[ z = x+y + \sum_{n=2}^{\infty} \sum_{\text{length } w = n} g_w w, \]

where the symbol \( w \) is now used to denote a word in the letters \( x \) and \( y \), e.g.,

\[ w = x^2 y^3 x y^4. \]
Let us call this the associative presentation of $z$. The coefficient $g_w$ is Goldberg's coefficient on word $w$. Goldberg's formulas are as follows:

Define polynomials $G_s(t)$ in a real variable $t$ recursively by

$$G_1(t) = 1,$$

then

$$G_s(t) = \frac{1}{s!} \sum_{t=1}^{s/2} \binom{s}{t} (t(t-1)G_{s-t}(t)).$$

If $w$ is the word

$$s_1 s_2 \cdots s_m \ (x \lor y)^{s_m},$$

with positive exponents $s_1, \ldots, s_m$, then its coefficient is

$$g_w = \sum_{t=0}^{m'} \binom{m}{t} (t(t-1))^m G_{s_1}(t) \cdots G_{s_m}(t) dt.$$

Here

$$m' = \left[ \frac{m}{2} \right], \quad m'' = \left\lfloor \frac{m-1}{2} \right\rfloor,$$

where $\left[ \cdot \right]$ denotes greatest integer.

These remarkable formulas were obtained by Goldberg simply by carefully keeping track of how terms may combine when the series $z = \log(1 + (e^x e^y - 1))$ is worked out, where $e^x e^y$ is the product of the Taylor series for $e^x$ and $e^y$.

There are some consequences of Goldberg's formulas.

First, the recursion shows that the roots of $G_s(t)$ are real, strictly interior to the interval $[0,1]$, and are symmetric relative to $\frac{1}{2}$. A simple application of Rolle's theorem proves this.

Next, a simple calculation shows that for $t$ in $[0,1]$,

$$(t-(\frac{1}{2}-r))(t-(\frac{1}{2}+r))$$
lies between $1/4$ and $-1/4$, when $0 \leq r \leq \frac{1}{2}$. From this, it is easy to deduce that

$$|G_s(t)| \leq 2^{-(s-1)} \text{ for } t \in [0,1].$$

(Note: degree $G_s(t) = s-1$.) This estimate for $|G_s(t)|$ will be crucial in our calculations, as follows. From

$$g_w = \int_0^1 t^{m'} (t-1)^{m''} G_{s_1}(t) \cdots G_{s_m}(t) dt,$$

it follows that

$$|g_w| \leq \int_0^1 t^{m'} (1-t)^{m''} 2^{-(s_1 \cdots s_m - m)} dt$$

$$= 2^{-(n-m)} \int_0^1 t^{m'} (1-t)^{m''} dt,$$

where $n$ is the length of word $w$, and $m$ is the number of parts it contains. A simple induction evaluates the integral here:

$$\int_0^1 t^{a} (1-t)^{b} dt = \frac{a!b!}{(a+b+1)!}.$$

From this, one gets this estimate for $g_w$:

$$|g_w| \leq \frac{2^{-(n-m)}}{m^{m-1}}.$$

Next, the number of words $g_w$ of length $n$ having $m$ parts and starting with $x$ is $\binom{n-1}{m-1}$. This is just a counting calculation. Using it, we find that
\[ \sum |g_w| \leq 2 \cdot \frac{2^{-(n-m)}(n-1)}{m'} \binom{m-1}{m'}, \]

where the sum on the left extends over all words \( w \) of length \( n \) having \( m \) parts, and starting with either \( x \) or \( y \). To include all degree \( n \) words, we must sum over \( m \) from 1 to \( n \).

Now let us use an associative norm, that is a norm for which \( \| xy \| \leq \| x \| \| y \| \). Summing the moduli of all degree \( n \) terms in Goldberg's series, we get

\[
\sum_{w, \text{length } w = n} |g_w| \| w \| \leq 2M^n \sum_{m=1}^{n} \frac{2^{-(n-m)}(n-1)}{m'} \binom{m-1}{m'},
\]

\[
= 2 \cdot 2^{-n} M^n \sum_{m=1}^{n} 2^m \frac{(n-1)}{m'} \binom{m-1}{m'} \tag{3}
\]

The majorizing series for the Goldberg associative series therefore is

\[
\sum_{n=1}^{\infty} 2 \cdot 2^{-n} M^n \sum_{m=1}^{n} 2^m \frac{(n-1)}{m'} \binom{m-1}{m'} \tag{4}
\]

In spite of the clumsy appearance of the expression, worthwhile information can be obtained from it. In the denominator term, \( \frac{(m-1)}{m'} \) is the largest (middle) term in the expansion of \( (1+1)^{m-1} \), an expansion having \( m \) terms. Therefore

\[ m' \binom{m-1}{m'} \geq 2^{m-1}. \]

Inserting this, and using
\[ \sum_{m=1}^{n} \binom{n-1}{m-1} = 2^{n-1}, \]

we get the following beautifully simple majorizing series for \( \Sigma g_{ww} \):

\[ 2\sum_{n=1}^{\infty} M^n, \quad (5) \]

and this is finite if \( M < 1 \).

This means:

The Goldberg associative series converges absolutely inside the unit sphere in any norm compatible with associative multiplication.\[ \]

Unfortunately, no more can be obtained from the majorizing series (4) for the Goldberg associative presentation of \( z \) because Wasin So has proved that it diverges on the boundary of the unit sphere. (This is sharper than saying that (5) diverges for \( M = 1 \).) So this technique only gives radius 1 as the radius of convergence.

Next, dividing the degree \( n \) terms by \( n \), we obtain this Lie presentation of \( z \):

\[ z = x + y + \sum_{n=2}^{\infty} \sum_{w, \text{length } w=n} \frac{1}{n} g_{ww}[w]. \]

Here \([w]\) denotes the left standardized iterated commutator based on \( w \). If the estimating done above is done again for this series, using a norm compatible with Lie multiplication (the calculations are the same, just divide length \( n \) words by \( n \)), then the majorizing series is

\[ 2 \cdot \sum_{n=1}^{\infty} 2^{-n} M^{n-1} \sum_{m=1}^{n} \binom{n-1}{m-1} \]

and using the same estimate as before on the denominator leads to the following, again beautifully simple, majorizing series for Goldberg's commutator series,
\[2 \sum_{n=1}^{\infty} \frac{1}{n^2} M^n = -\log(1 - M),\]

and this is finite if \( M < 1 \). In this new calculation we use the compatibility with Lie multiplication:

\[\|[u, v]\| \leq \|u\| \|v\|.\]

This proves:

The Goldberg commutator series for \( z \) converges absolutely in the interior of the unit sphere in any norm compatible with Lie multiplication.

Wasin So proved that the majorizing series (6) with the factor \( 1/n \) on the degree \( n \) terms in fact has the closed unit sphere for its radius of convergence. So passing to commutators yields absolute convergence on the boundary of the unit sphere, but nothing more.

There is another and rather different convergence proof in the well known book *Lie groups and Lie algebras* by V. S. Varadarajan. It goes this way. First write \( z \) as a sum of homogeneous terms,

\[z = x + y + \sum_{n=2}^{\infty} G_n(x, y)\]

where \( G_n(x, y) \) is the degree \( n \) homogeneous component of \( z \), namely the sum of all degree \( n \) terms. Let us call this the presentation of \( z \) in homogeneous components. The theorem in Varadarajan is that, under a Lie norm, we have

\[\|z\| \leq \|x\| + \|y\| + \sum_{n=2}^{\infty} \|G_n(x, y)\| < \infty\]

if \( \|x\| < \frac{\delta}{2} \) and \( \|y\| < \frac{\delta}{2} \), where \( \delta \) is defined as follows. Expand (Taylor)
\[ \frac{\zeta}{1-e^{-\zeta}} - \frac{1}{2}\zeta = 1 + \sum_{p=1}^{\infty} K_{2p} \zeta^{2p}. \]

Here \( K_{2p} = B_{2p}/(2p)! \), \( B_{2p} \) being the usual Bernoulli number, and \( \zeta \) is a complex variable. Now define function \( H(\zeta) \) as

\[ H(\zeta) = 1 + \sum_{p=1}^{\infty} |K_{2p}| \zeta^{2p}. \]

Then \( H(\zeta) \) is analytic in \(|\zeta| < 2\pi\). Next, define \( y = y(t) \) as the solution of this differential equation:

\[ \frac{dy}{dt} = \frac{1}{2} y + H(y), \quad y(0) = 0. \]

This is to be a differential equation in the complex domain. By differential equation theory, the solution function \( y(t) \) is analytic in some disk of radius \( \delta \) centered at the origin. This \( \delta \) is the number that appears in the convergence theorem for the exponent \( z \), namely the series for \( z \) in homogeneous components converges for \( x, y \) normed of radius \( < \frac{1}{2} \delta \) (using a Lie norm.) Note the factor \( \frac{1}{2} \).

And now the question arises: What is \( \delta \)?

It is already proved in Varadarajan's book that the series giving the solution function \( y(t) \) has all coefficients positive. Therefore, by a theorem in complex analysis, the singularity of the solution \( y(t) \) nearest the origin occurs on the positive real axis. Therefore the search for \( \delta \) may be confined to the real case.

I programmed and ran a Runge Kutta algorithm on my IBM XT compatible computer to get the solution \( y(t) \). Specifically, I used the 4th order Runge Kutta Fehlberg algorithm with error control and adaptive step size, and I found that the program steadily reduced its step size to the preset minimum of \( 10^{-9} \), at which point it bombed, at \( t = 2.1737 \cdots \), with the value of \( y(t) \) for this \( t \) almost exactly \( 6.283285 \cdots \).

What are these numbers? I was so pleased that I got the program to work that I didn't look closely at the numerical output. The \( 2.1737 \cdots \) is not recognizable, but the
is. It is ??????????? So, something is going on. I didn't recognize this value, but Morris Newman identified it instantly, and understood its meaning.

The Bernoulli numbers are well known to alternate in sign. This means that in

\[ H(\zeta) = 1 + \sum_{p=1}^{\infty} |K_{2p}| \zeta^{2p} \]

one should attempt to replace \( \zeta \) by \( i\zeta \). Indeed, using the known signs of the Bernoulli numbers, we see that

\[ H(\zeta) = 1 + \sum_{p=1}^{\infty} |K_{2p}| i^{2p} = 2 - (1 + \sum_{p=1}^{\infty} K_{2p} (i\zeta)^{2p}) \]

\[ = 2 - \left( \frac{i\zeta}{1 - e^{-i\zeta}} - \frac{i}{2} i\zeta \right) \]

\[ = 2 - \frac{\zeta}{2} \cot \frac{\zeta}{2} \]

Thus the differential equation for function \( y(t) \) is

\[ \frac{dy}{dt} = 2 + \frac{1}{2} y - \frac{1}{2} y \cot \frac{1}{2} y, \quad \text{with } y(0) = 0. \]

The right side is analytic for \( y \) in the interval \([0,2\pi)\). If we rewrite the differential equation as

\[ \frac{dt}{dy} = \frac{1}{2 + \frac{1}{2} y - \frac{1}{2} y \cot \frac{1}{2} y}, \quad t(0) = 0, \]

then we see that \( t \) is analytic in \( y \) for \( y \) in \([0,2\pi)\). The inverse function \( y(t) \) will therefore be analytic in \( t \) for \( t \) in the interval \([0,?]\), where ? is the \( t \) value corresponding to \( y = 2\pi \). This means that the \( t \) value, \( \delta \), producing the singularity nearest 0 is
\[
 t = \int_0^{2\pi} \frac{1}{2 + \frac{1}{2} y - \frac{1}{2} y \cot \frac{1}{2} y} \, dy.
\]

Morris Newman ran a Simpson's rule quadrature on this, using many nodes, and obtained the same value \(2.1737\ldots\) as I got from my Runge Kutta Fehlberg computation.

Therefore \(\delta = 2.1737\ldots\), and so \(\frac{1}{2}\delta = 1.08685\ldots\).

This means: if a norm compatible with Lie multiplication is used, and if the exponent \(z\) is regarded as a sum of homogeneous terms,

\[
 z = x + y + \sum_{n=2}^{\infty} G_n(x,y),
\]

then the series converges in a sphere around the origin just slightly larger than the unit sphere.

However, there is a remarkably simple argument that yields a bigger convergence sphere. In (7) each \(G_n(x,y)\) is a Lie element, a linear combination of commutators. But let us write it in noncommutator form (expand the commutators.) Then, using a Lie norm derived from an associative norm, we have

\[
 \|z\| \leq \|x\| + \|y\| + \sum_{n=2}^{\infty} \|G_n(x,y)\|.
\]

Substitute a sum of Goldberg terms for each \(G_n(x,y)\), and then expand. We therefore get

\[
 \leq \|x\| + \|y\| + \sum_{n=2}^{\infty} \sum_{\text{length } w = n} |g_{w!}w| w, < \infty
\]

provided \(\|x\|, \|y\| < 1\) in the associative norm. But

\[
 \| \|_{\text{Lie}} = 2 \| \|_{\text{associative}},
\]

so that \(x, y\) being \(< 1\) in an associative norm means being \(< 2\) in a Lie norm. Thus the convergence bound \(\frac{1}{2}\delta = 1.08\ldots\) from Varadarajan's
book has been increased to the sphere of norm 2 centered on the origin.

This suggests that the factor $\frac{1}{2}$ in the $\frac{1}{2} \delta$ of Varadarajan's theorem is not needed, and that the series for the exponent $z$ converges in the sphere centered on 0 with radius $\delta = 2.1737\cdots$. This is not yet proved.

Thus, falling back on the associative series can bring stronger results than those previously known for the Lie series for the exponent $z$.

Undoubtedly, these bounds are too small and convergence occurs in larger regions.

A table will help to summarize this information:

(i) Dynkin Lie series:
    converges for $\|x\| < .34657\cdots, \|y\| < .34657\cdots$, under a Lie norm.

(ii) Goldberg associative series:
    converges in the open unit sphere about 0, under an associative norm.

(iii) Goldberg Lie series:
    converges in the closed unit sphere about 0, under a Lie norm.

(iv) Presentation of $z$ in homogeneous components:
    converges in the open sphere of radius 1.08685\cdots, under a Lie norm.

(v) Presentation of $z$ in homogeneous components:
    converges in an open sphere about 0 of radius 2, under a Lie norm derived from an associative norm.

(vi) Presentation of $z$ in homogeneous components:
    conjectured to converge in an open sphere about 0 of radius 2.1737\cdots, under a Lie norm.
There are analytic formulas known for the exponent \( z \). Here is one:

\[
    z = x + \int_0^1 \psi((e^{ad}x)(e^{t\delta y}))(x)\,dt,
\]

where \( adu \) is the linear operator for which \((adu)(v) = uv - vu\), and \( \psi \) is the function defined by

\[
    \psi(\zeta) = \frac{\zeta \log \zeta}{\zeta - 1}.
\]

This formula presumably carries all knowledge of the convergence of the series for exponent \( z \), however, extracting this knowledge seems difficult.

Another question that arises here is this: is every Lie norm on a Lie algebra derived from an associative norm on the enveloping associative algebra? (There always is an enveloping associative algebra, a well known fact in Lie theory.)

**Part II**

The matrix valued numerical range

This will be a totally unrelated topic in which the number 2 also appears to play a central role.

Let \( A \) be a complex matrix. The numerical range of \( A \) is the set of all \( x^*Ax \) as \( x \) ranges over all complex \( n \) vectors. This is a subset of the complex plane, and the famous Toeplitz-Hausdorff theorem asserts that it is a convex subset. Another way of looking at the numerical range is as the collection of \((1,1)\) elements of \( UAU^* \) as \( U \) ranges over all \( n \times n \) unitary matrices.

I have long been interested in the larger principal submatrices of \( UAU^* \) as \( U \) ranges over unitary matrices. That is, I wish study the set of \( k \times k \) leading principal submatrices of \( UAU^* \) as \( U \) ranges over all unitary matrices and \( k \) is fixed. Let us call this the \( k\)th order matrix numerical range of \( n \times n \) matrix \( A \).
But no one to whom I mentioned this topic seemed interested in it. So, I eventually began a closer look at it myself, and almost immediately found a theorem. The obvious thing to do as a first step is to go to the opposite extreme from the conventional case, that is set $k = n$, ($k = 1$ is the Toeplitz Hausdorff theorem.) So we consider the set of all $n \times n$ matrices of the form $UAU^*$, as $U$ varies over all $n \times n$ unitary matrices, and $A$ is fixed. We regard this set as lying in $n^2$ dimensional complex space, which we then interpret as $2n^2$ real space. And in $2n^2$ real space, we ask: Is this set of matrices a convex set? The answer is no, and in fact an extreme form of nonconvexity holds:

The $n^{th}$ order matrix numerical range never has three collinear points.

Thus, the lowest numerical range, for $k = 1$, is always convex, and the highest, for $k = n$, never has three collinear points. What happens for the intermediate values of $k$? I posed this as a research problem in Linear and Multilinear Algebra. N. Tsing and C. K. Li have started to work on it. Their results are very preliminary, but seem to point in this direction: for $k < \frac{1}{2}n$, there is some hint of convexity, for bigger $k$, none. If the set isn't convex, does it have any worthwhile properties? The question is open.

An obvious tool to apply is Specht's theorem stating that two matrices $A$ and $B$ are unitarily similar if and only if each word in $A$ and $A^*$ has the same trace as the corresponding word in $B$ and $B^*$. Finitely many words are known to suffice. But no one appears to have applied Specht's theorem to numerical range questions in a significant way.

The proof of my nonconvexity theorem is very simple. Given distinct matrices $L_0$ and $L$ in the $n^{th}$ order numerical range, the real line containing them is the set of all

$$L_0 + rL_1$$

as $r$ ranges over real numbers, where $L_1 = L - L_0$. We ask: for how many values of $r$ do we have

$$L_0 + rL_1 = UAU^*$$
for some unitary $U$, dependent on $r$? This equation implies

$$\text{tr}(L_0 + rL_1)(L_0 + rL_1)^* = \text{tr}AA^*,$$

and this is a quadratic equation in $r$, with the coefficient of $r^2$ nonzero. So it can be satisfied by at most two values of $r$, completing the proof.

Since this proof used a trace argument, it suggests that the technique can be extended by invoking Specht's theorem. However, Specht's theorem applies only to unitary similarity, and not to submatrices of a unitary similarity transform of a given matrix. I attempted to adjust this argument, without success. Other ideas will be needed.

The classical numerical range has been the source of many good theorems for many mathematicians. I believe the matrix numerical ranges should be at least equally fertile, and perhaps even more so. The area is wide open, with almost nothing proved.

End of lecture 10

I hope you got something worthwhile from these lectures.