Notes on a theorem of Silver's

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Silver's Theorem. *Relative to the consistency of a cardinal κ that is κ^{++}-supercompact, it is consistent that there exists a measurable cardinal κ such that $2^\kappa = \kappa^{++}$.*

A theorem of Kunen's [**J**, *pp.* 450*ff.*] states that if the GCH fails at a measurable cardinal, then it is consistent that there exist α many measurable cardinals, for each ordinal α. Thus some hypothesis stronger than measurability is necessary to obtain the consistency of the GCH failing at a measurable cardinal.

Furthermore, if the GCH fails at a measurable cardinal, then it must fail at many smaller inaccessible cardinals.

Silver originally proved his theorem using a backwards Easton support iteration of Cohen product forcing that adds α^{++} many subsets to each inaccessible α less than or equal to κ. The purpose of this note is to give an alternate proof that avoids such a backwards Easton forcing in favor of a forwards Easton forcing that adds a κ-tree with κ^{++} many branches while preserving the measurability of κ. Only this part of the proof is different from Silver's; our use of supercompactness is essentially the same.

The forwards Easton forcing

For inaccessible cardinals α, set $T_\alpha = \{\alpha\} \times \alpha^{++}$. Let α^* denote the least inaccessible cardinal greater than α, provided that there exists an inaccessible greater than α.

Fix an inaccessible cardinal κ and declare that $p \in \mathbb{P}^\kappa$ iff p is a function and

(1) the domain of p is an Easton set of inaccessible cardinals less than κ.

Call the domain of p its *support* and let $\mathrm{sp}(p)$ denote it. In order that p be a condition in $\mathbb{P}^\kappa$, also require that

(2) if α lies in the support of p, then $p_\alpha\colon T_\kappa \to T_\alpha$ and $|p_\alpha| < \alpha^*$;
(3) if $\alpha < \beta$ lie in the support of p, then $\mathrm{dom}(p_\alpha) \subseteq \mathrm{dom}(p_\beta)$; and
(4) (no M's) if $\alpha < \beta$ lie in the support of p and $u, u' \in \mathrm{dom}(p_\alpha)$ and if $p_\beta(u) = p_\beta(u')$, then $p_\alpha(u) = p_\alpha(u')$.

Declare that $\bar{p} \geqslant p$ iff $\mathrm{sp}(\bar{p}) \subseteq \mathrm{sp}(p)$ and $\bar{p}_\alpha \subseteq p_\alpha$, for each $\alpha \in \mathrm{sp}(\bar{p})$.

The motivation for this definition is that each u in T_κ should index a branch through a generic κ-tree, specifically that $p_\alpha(u) = t$ commits t to lie in the branch indexed by u.

In the constructions that follow, properties (1)–(3) will be uncontroversial; only (4) will require verification. Call objects satisfying properties (1)–(3) *pseudo-conditions*. Then to verify that a pseudo-condition is a condition, we must check that it has no M's.

This is version 1.1.

If p is a pseudo-condition, set $r(p) = \bigcup_{\alpha \in \mathrm{sp}(p)} \mathrm{rng}(p_\alpha)$ and define a relation $\dashv_{p,i}$ on $r(p)$ by

$$t \dashv_{p,i} t' \quad \text{iff} \quad \begin{array}{l} t \in T_\alpha \text{ and } t' \in T_\beta, \text{ for some } \alpha < \beta, \text{ and there exists} \\ \text{a } u \in T_\kappa \text{ such that } p_\alpha(u) = t \text{ and } p_\beta(u) = t'. \end{array}$$

That is, $t \dashv_{p,i} t'$ when $t \in T_\alpha$ and $t' \in T_\beta$, for some $\alpha < \beta$, and t and t' have a common preimage under p.

Let $\dashv_p$ be the transitive closure of $\dashv_{p,i}$. Then $t \dashv_p t'$ iff there exists a finite sequence $t_0, \ldots, t_n$ such that $t = t_0 \dashv_{p,i} \cdots \dashv_{p,i} t_n = t'$.

Clearly $\dashv_p$ is a well-founded partial ordering.

Lemma. *Suppose that p is a pseudo-condition. Then $\dashv_p$ is a tree ordering iff p is a condition.*

Proof: ($\Leftarrow$) Suppose that $\dashv_p$ is not a tree ordering. Then the $\dashv_p$-predecessors of some t are not linearly ordered. Choose α to be least such that there exists $t \in T_\alpha$ with $t', t'' \dashv_p t$ and t' not comparable to t'' in $\dashv_p$. Then we may assume that $t', t'' \dashv_{p,i} t$. Say $t' \in T_{\alpha'}$ and $t'' \in T_{\alpha''}$ and $\alpha'' \leqslant \alpha'$. Let $u' \in T_\kappa$ be a common preimage under p of t and t', and let u'' be a common preimage under p of t and t''. Now $u'' \in \mathrm{dom}(p_{\alpha'})$ and $p_{\alpha'}(u'') \neq t'$, or else u'' is a common preimage under p of both t'' and t', contradicting that $t'' \not\dashv_p t'$. But then $p_\alpha(u') = t = p_\alpha(u'')$ and $p_{\alpha'}(u'') \neq t' = p_{\alpha'}(u')$ and so p is not a condition.

($\Rightarrow$) Suppose that $\alpha < \beta$ lie in the support of p, that $u, u' \in \mathrm{dom}(p_\alpha)$, and that $p_\beta(u) = p_\beta(u')$, but that $p_\alpha(u) \neq p_\alpha(u')$. Then $p_\alpha(u), p_\alpha(u') \dashv_p p_\beta(u) = p_\beta(u')$, but $p_\alpha(u)$ and $p_\alpha(u')$ are not $\dashv_p$-comparable. □

Extension Lemma. *Suppose that $\bar{p} \in \mathbb{P}^\kappa$.*

(a) *If $\beta < \kappa$ is inaccessible, then there exists a condition p extending $\bar{p}$ with $\beta \in \mathrm{sp}(p)$ and*

(b) *if $\beta \in \mathrm{sp}(\bar{p})$ and $u \in T_\kappa$, then there exists a condition p extending $\bar{p}$ such that $u \in \mathrm{dom}(p_\beta)$ and $\mathrm{sp}(p) = \mathrm{sp}(\bar{p})$.*

Proof of (a): Suppose that $\beta \notin \mathrm{sp}(\bar{p})$. Set $U = \bigcup_{\alpha \in \mathrm{sp}(\bar{p}) \cap \beta} \mathrm{dom}(\bar{p}_\alpha)$. For $u, u' \in U$, declare that $u \sim u'$ iff $\bar{p}_\alpha(u) = \bar{p}_\alpha(u')$, for all sufficiently large $\alpha \in \mathrm{sp}(\bar{p}) \cap \beta$. Then $\sim$ is an equivalence relation on U. Now $|U| < \beta$, so there exists a function $f : U \to T_\beta$ such that $f(u) = f(u')$ iff $u \sim u'$. Let p be identical with $\bar{p}$, except that $\beta \in \mathrm{sp}(p)$ and $p_\beta = f$. Then p is a pseudo-condition.

Suppose that $\delta < \gamma$ lie in $\mathrm{sp}(p)$, that $u, u' \in \mathrm{dom}(p_\delta)$, and that $p_\gamma(u) = p_\gamma(u')$. We must argue that $p_\delta(u) = p_\delta(u')$. Since $\bar{p}$ is a condition, we may assume that $\delta = \beta$ or $\gamma = \beta$.

If $\delta = \beta$, then $u, u' \in U$, hence $u, u' \in \mathrm{dom}(p_\alpha)$, for all sufficiently large $\alpha \in \mathrm{sp}(\bar{p}) \cap \beta$. Furthermore, $\bar{p}_\alpha(u) = \bar{p}_\alpha(u')$, for all such α, because $\bar{p}_\gamma(u) = \bar{p}_\gamma(u')$. Hence $u \sim u'$ and $p_\beta(u) = p_\beta(u')$.

If $\gamma = \beta$, then $p_\beta(u) = p_\beta(u')$. Hence $u \sim u'$. It follows that $\bar{p}_\alpha(u) = \bar{p}_\alpha(u')$, for some α such that $\delta \leqslant \alpha < \beta$, and so that $p_\delta(u) = p_\delta(u')$.

Proof of (b): Fix $u \in T_\kappa$ and suppose that $u \notin \mathrm{dom}(\bar{p}_\beta)$.

Note first that we may assume that $\mathrm{dom}(\bar{p}_\alpha) \neq \emptyset$, for some $\alpha \in \mathrm{sp}(\bar{p})$. Otherwise, we could choose $t_\alpha \in T_\alpha$, for each $\alpha \in \mathrm{sp}(\bar{p})$, and extend $\bar{p}$ to obtain p with $\mathrm{dom}(p_\alpha) = \{u\}$ and $p_\alpha(u) = t_\alpha$, for each $\alpha \in \mathrm{sp}(\bar{p}) = \mathrm{sp}(p)$.

In fact, we may assume that $\mathrm{dom}(\bar{p}_\alpha) \neq \emptyset$, for all $\alpha \in \mathrm{sp}(\bar{p})$. Otherwise, we could choose α to be least such that $\mathrm{dom}(\bar{p}_\alpha) \neq \emptyset$, choose any $u' \in \mathrm{dom}(\bar{p}_\alpha)$ and any $t_\gamma \in T_\gamma$, for $\gamma \in \mathrm{sp}(\bar{p}) \cap \alpha$, and let p extending $\bar{p}$ be identical with p, except that for $\gamma \in \mathrm{sp}(\bar{p}) \cap \alpha$, $\mathrm{dom}(p_\gamma) = \{u'\}$ and $p_\gamma(u') = t_\gamma$.

It follows that we may further assume that $u \in \mathrm{dom}(\bar{p}_\alpha)$, for some $\alpha \in \mathrm{sp}(\bar{p})$. If not, we could choose any $u' \in \mathrm{dom}(\bar{p}_\alpha)$, where α is least in $\mathrm{sp}(\bar{p})$, and let p extending $\bar{p}$ be identical with $\bar{p}$, except that $\mathrm{dom}(p_\gamma) = \mathrm{dom}(\bar{p}_\gamma) \cup \{u\}$ and $p_\gamma(u) = \bar{p}_\gamma(u')$, for all $\gamma \in \mathrm{sp}(\bar{p})$.

Let us now fix $\alpha \in \mathrm{sp}(\bar{p})$ to be least such that $u \in \mathrm{dom}(\bar{p}_\alpha)$. Note that we may assume that $\bar{p}_\alpha(u)$ is $\dashv_{\bar{p}}$-minimal. Indeed, if there exists $\bar{\alpha} < \alpha$ such that $\bar{p}_\alpha(u)$ has a $\dashv_{\bar{p}}$-predecessor in $T_{\bar{\alpha}}$, then, letting p be identical with $\bar{p}$, except that $u \in \mathrm{dom}(p_\gamma)$ and $p_\gamma(u)$ is the $t \in T_\gamma$ such that $t \dashv_{\bar{p}} \bar{p}_\alpha(u)$, for each $\gamma \in \mathrm{sp}(\bar{p}) \cap [\bar{\alpha}, \alpha)$, we obtain a pseudo-condition p such that $\dashv_p$ is identical with $\dashv_{\bar{p}}$. It follows that p is a condition.

So let us suppose that $\bar{p}_\alpha(u)$ is $\dashv_{\bar{p}}$-minimal. Choose any $u' \in \mathrm{dom}(\bar{p}_\beta)$ and let p be identical with $\bar{p}$, except that $p_\gamma(u) = \bar{p}_\gamma(u')$, for $\gamma \in \mathrm{sp}(\bar{p}) \cap [\beta, \alpha)$. Then p is a pseudo-condition and

$$\dashv_p = \dashv_{\bar{p}} \cup \Big\{ (t', t) : \bar{p}_\alpha(u) \underline{\dashv}_{\bar{p}}\, t \text{ and } t' \underline{\dashv}_{\bar{p}}\, \bar{p}_\gamma(u'), \text{ for some } \gamma \in \mathrm{sp}(\bar{p}) \cap [\beta, \alpha) \Big\}$$

Using that $\dashv_{\bar{p}}$ is a tree ordering, it can be seen that $\dashv_p$ is as well. Hence p is as required. □

Set $T = \bigcup_{\alpha<\kappa} T_\alpha$ and suppose that G is a filter on $\mathbb{P}^\kappa$. Set $\dashv_G = \bigcup_{p\in G} \dashv_p$. Then $\dashv_G$ is a tree ordering on T.

Below we shall argue that $\mathbb{P}^\kappa$ is cardinal preserving and calculate cardinal exponents in $\mathbb{P}^\kappa$ generic extensions. The following proposition simply remarks that $\mathbb{P}^\kappa$ conforms to its motivation.

Tree Lemma. *Suppose that κ is an inaccessible limit of inaccessibles and that G is $\mathbb{P}$ generic over V. Then $(T, \dashv_G)$ is a κ-tree with κ^{++} many branches.*

Proof: Using that G is generic and that κ is a limit of inaccessibles, it is easy to see that $(T, \dashv_G)$ has height κ. Using that $|T_\alpha| = \alpha^{++} < \kappa$, for inaccessible $\alpha < \kappa$, it follows that $(T, \dashv_G)$ is a κ-tree.

For $u \in T_\kappa$, set

$$b_u = \big\{\, p_\alpha(u) : p \in G \text{ and } \alpha \in \mathrm{sp}(p) \text{ and } u \in \mathrm{dom}(p_\alpha) \,\big\}.$$

Suppose that $u \neq u'$ lie in T_κ and that $u, u' \in d(p) = \bigcup_{\alpha\in\mathrm{sp}(\bar{p})} \mathrm{dom}(\bar{p}_\alpha)$, for some $\bar{p} \in \mathbb{P}^\kappa$. Choose an inaccessible β such that $\sup\big(\mathrm{sp}(\bar{p})\big) < \beta < \kappa$. Now $|d(\bar{p})| < \sup\big(\mathrm{sp}(\bar{p})\big)^* \leqslant \beta$, so there exists a one-to-one function $f\colon d(\bar{p}) \to T_\beta$. Let p be identical with $\bar{p}$, except that $\mathrm{sp}(p) = \mathrm{sp}(\bar{p}) \cup \{\beta\}$ and $p_\beta = f$. Then p is a condition extending $\bar{p}$ and $p \Vdash b_u \neq b_{u'}$. □

Antichain Lemma. *Suppose that κ is inaccessible, but not the least inaccessible cardinal. Set $\lambda = \sup\{\, \mu^* : \mu < \kappa$ is inaccessible$\,\}$. Then $\mathbb{P}^\kappa$ has antichains of cardinality at most $\lambda^{<\lambda}$.*

If κ is the least inaccessible, then $\mathbb{P}^\kappa = \{\emptyset\}$.

If κ is inaccessible and is greater than the least inaccessible cardinal, then either $\lambda = \kappa$, or $\lambda < \kappa$ is a singular strong limit cardinal. In the former case $\lambda^{<\lambda} = \kappa$; in the latter, $2^\lambda \leqslant \lambda^{\mathrm{cf}(\lambda)}$, so $\lambda^{<\lambda} = \lambda^\lambda = 2^\lambda = \lambda^{\mathrm{cf}(\lambda)}$.

Proof: Suppose that $A \subseteq \mathbb{P}^\kappa$ has cardinality $(\lambda^{<\lambda})^+$. We must see that A is not an antichain. Noting that λ has $\lambda^{<\lambda}$ many Easton subsets, we may assume that there exists a fixed $D \subseteq \lambda$ such that $\mathrm{sp}(p) = D$, for all $p \in A$. As previously, set $d(p) = \bigcup_{\alpha\in D} \mathrm{dom}(p_\alpha)$. We may assume that $\mathcal{F} = \{\, d(p) : p \in A \,\}$ forms a Δ-system. (If $\lambda = \kappa$, then without loss of generality, $\mathcal{F}$ is a family of κ^+ many sets, each of cardinality less than κ; if $\lambda < \kappa$, then, without loss of generality, $\mathcal{F}$ is a family of $(2^\lambda)^+$ many sets, each of cardinality less than λ^+.) Say the root of $\mathcal{F}$ is r. We may assume that $\mathrm{rng}(p_\alpha) = \mathrm{rng}(p'_\alpha)$, for all $p, p' \in A$ and all $\alpha \in D$. Finally, if $p, p' \in A$, we may assume that there exists a bijection $e\colon d(p) \to d(p')$ such that

- $e \restriction r = \mathrm{id} \restriction r$;
- $u \in \mathrm{dom}(p_\alpha)$ iff $e(u) \in \mathrm{dom}(p'_\alpha)$, for all $\alpha \in D$ and $u \in d(p)$; and
- $p_\alpha(u) = p'_\alpha\big(e(u)\big)$, for all $\alpha \in D$ and $u \in \mathrm{dom}(p_\alpha)$.

(Consider isomorphism types of $p \in A$ in a language with function symbols for each p_α and constant symbols for each element of r and of $\bigcup_{\alpha\in D} \mathrm{rng}(p_\alpha)$.) Call such an e an *isomorphism* from p to p'.

Suppose that $p, p' \in A$. Let $\mathrm{sp}(q) = D$ and set $q_\alpha = p_\alpha \cup p'_\alpha$, for $\alpha \in D$. We maintain that q is a condition extending both p and p'. To see that when $\alpha \in D$, the set q_α is a function, let e be an isomorphism from p to p'. If $u \in \mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p'_\alpha)$, then $u \in r$ and so $p_\alpha\big(u\big) = p'_\alpha\big(e(u)\big) = p'_\alpha\big(u\big)$. Thus q is a pseudo-condition.

Suppose now that $\alpha < \gamma$ lie in D, that $u, u' \in \mathrm{dom}(q_\alpha)$, and that $q_\gamma(u) = q_\gamma(u')$. We must see that $q_\alpha(u) = q_\alpha(u')$. We may assume that $u \in \mathrm{dom}(p_\gamma)$ and $u' \in \mathrm{dom}(p'_\gamma)$. Let e be an isomorphism from p to p'. Then $p_\gamma(u) = p'_\gamma(u')$, so $p'_\gamma\big(e(u)\big) = p'_\gamma(u')$. But then $p'_\alpha\big(e(u)\big) = p'_\alpha\big(u'\big)$. Hence $p_\alpha(u) = p'_\alpha(u')$. $\square$

Suppose that $\mu < \kappa$ and μ is inaccessible. Set

$$\mathbb{P}^\kappa_\mu = \big\{\, p \in \mathbb{P}^\kappa : \mathrm{sp}(p) \cap \mu = \emptyset \,\big\}.$$

Then $\mathbb{P}^\kappa_\mu$ is $<\mu^*$-closed.

Factor Lemma. *Suppose that $\mu < \kappa$ are inaccessible. Then $\mathbb{P}^\kappa$ is equivalent to the product $\mathbb{P}^\mu \times \mathbb{P}^\kappa_\mu$.*

Proof: It suffices to define a function

$$e\colon \Big\{\, p \in \mathbb{P} : \mu \in \mathrm{sp}(p) \,\Big\} \to \mathbb{P}^\mu \times \mathbb{P}^\kappa_\mu$$

such that

- $\bar{p} \geqslant p$ iff $e(\bar{p}) \geqslant e(p)$, and
- the range of e is dense in $\mathbb{P}^\mu \times \mathbb{P}^\kappa_\mu$.

Suppose that $p \in \mathbb{P}^\kappa$ and that $\mu \in \mathrm{sp}(p)$. Set $e(p) = (q, r)$, where $r = p \restriction [\mu, \kappa)$ and q is defined as follows: Set $\mathrm{sp}(q) = \mathrm{sp}(p) \cap \mu$. For $\alpha \in \mathrm{sp}(q)$, set $\mathrm{dom}(q_\alpha) = p_\mu\text{''}\,\mathrm{dom}(p_\alpha)$, which has cardinality less than α^*. If $t \in \mathrm{dom}(q_\alpha)$, set $q_\alpha(t) = p_\alpha(u)$, where $u \in p_\mu^{-1}(t) \cap \mathrm{dom}(p_\alpha)$. A key observation is that this definition of $q_\alpha(t)$ does not depend on our choice of u. Indeed, if $u, u' \in \mathrm{dom}(p_\alpha)$ and $p_\mu(u) = p_\mu(u') = t$, then $p_\alpha(u) = p_\alpha(u')$. This completes the definition of e.

Assume first that $\bar{p} \geqslant p$. Set $e(\bar{p}) = (\bar{q}, \bar{r})$ and $e(p) = (q, r)$. We must see that $(\bar{q}, \bar{r}) \geqslant (q, r)$. Certainly $\bar{r} \geqslant r$. Suppose, then, that $t \in \mathrm{dom}(\bar{q}_\alpha)$. Say $\bar{p}_\mu(u) = t$. Then $p_\mu(u) = t$, so $\bar{q}_\alpha(t) = \bar{p}_\alpha(u) = p_\alpha(u) = q_\alpha(t)$. Hence $\bar{q} \geqslant q$, as well.

Conversely, suppose that $(\bar{q}, \bar{r}) \geqslant (q, r)$, where $e(\bar{p}) = (\bar{q}, \bar{r})$ and $e(p) = (q, r)$. We must see that $\bar{p}_\alpha \subseteq p_\alpha$, for all $\alpha \in \mathrm{sp}(\bar{p})$. This is clear if $\alpha \geqslant \mu$, since $\bar{r} \geqslant r$. If $\alpha < \mu$ and $u \in \mathrm{dom}(\bar{p}_\alpha)$, set $t = \bar{p}_\mu(u)$. Then $t \in \mathrm{dom}(\bar{q}_\alpha)$ and $\bar{q}_\alpha(t) = \bar{p}_\alpha(u)$. Since $\bar{p}_\mu = \bar{r}_\mu \subseteq r_\mu = p_\mu$, also $q_\alpha(t) = p_\alpha(u)$. And $\bar{q}_\alpha(t) = q_\alpha(t)$, since $\bar{q}_\alpha \subseteq q_\alpha$.

Finally, to see that the range of e is dense in $\mathbb{P}^\mu \times \mathbb{P}^\kappa_\mu$, note first that the collection of pairs (q, r) such that $\mu \in \mathrm{sp}(r)$ and $\mathrm{dom}(q_\alpha) \subseteq \mathrm{rng}(r_\mu)$, for all $\alpha \in \mathrm{sp}(q)$, is dense in $\mathbb{P}^\mu \times \mathbb{P}^\kappa_\mu$. This uses the Extension Lemma, that $|d(q)| < \mu$, and that $\mathbb{P}^\kappa_\mu$ is $<\mu^*$-closed. If (q, r) is such a pair, then define $p \in \mathbb{P}^\kappa$ with $\mathrm{sp}(p) = \mathrm{sp}(q) \cup \mathrm{sp}(r)$ as follows: If $\alpha \in \mathrm{sp}(r)$, set $p_\alpha = r_\alpha$. If $\alpha \in \mathrm{sp}(q)$, for each $t \in \mathrm{dom}(q_\alpha)$, choose $u_t \in \mathrm{dom}(r_\mu)$ such that $r_\mu(u_t) = t$. Set $\mathrm{dom}(p_\alpha) = \{\, u_t : t \in \mathrm{dom}(q_\alpha) \,\}$, which has cardinality less than α^*, and set $p_\alpha(u_t) = q_\alpha(u)$. Then $e(p) = (q, r)$. □

Cardinal Preservation Lemma. *Assume the GCH in the ground model. If κ is inaccessible and α is a regular cardinal, then in a $\mathbb{P}^\kappa$ generic extension the range of each α-sequence is covered by a ground model set of ground model cardinality α. Consequently, $\mathbb{P}^\kappa$ is cardinal preserving.*

Proof: Fix α. Since $\mathbb{P}^\kappa$ satisfies the $<\kappa^+$-chain condition, the lemma is clear if $\alpha \geqslant \kappa$. Suppose that $\alpha < \kappa$.

Case 1. There exists a largest inaccessible $\mu \leqslant \alpha$. Then $\mathbb{P}^\kappa$ is equivalent to $\mathbb{P}^\mu \times \mathbb{P}^\kappa_\mu$. Now $\mathbb{P}^\kappa_\mu$ is $\leqslant\alpha$-closed, because $\alpha < \mu^*$, and $\mathbb{P}^\mu$ has antichains of size at most μ. Our claim follows.

Case 2. There does not exist a largest inaccessible less than or equal to α. Set $\lambda = \sup\{\, \nu < \alpha : \nu \text{ is inaccessible} \,\} = \{\, \nu^* < \alpha : \nu \text{ is inaccessible} \}$. Then λ is singular or $\lambda = 0$, so $\lambda < \alpha$. Let μ be the least inaccessible greater than α. Now $\mathbb{P}^\kappa$ is equivalent to $\mathbb{P}^\mu \times \mathbb{P}^\kappa_\mu$. And $\mathbb{P}^\kappa_\mu$ is $\leqslant\alpha$-closed; and if $\mathbb{P}^\mu$ is non-trivial, then $\mathbb{P}^\mu$ has antichains of size at most $\lambda^{<\lambda} = \lambda^+ \leqslant \alpha$. □

Cardinal Exponentiation Lemma. *Assume the GCH in the ground model V. For inaccessible μ, set*

$$\lambda_\mu = \sup\{\, \nu^* : \nu < \mu \text{ is inaccessible} \}.$$

Suppose that κ is inaccessible, that G is $\mathbb{P}^\kappa$ generic, and that α is an infinite cardinal greater than the least inaccessible. Then $2^\alpha = \mu^{++}$ in $V[G]$, if $\alpha = \mu$ or if $\lambda_\mu < \alpha < \mu$, for some inaccessible $\mu \leqslant \kappa$. Otherwise $2^\alpha = \alpha^+$.

Proof: If $\alpha > \kappa$, then $2^\alpha = \alpha^+$, since $|\mathbb{P}^\kappa| = \kappa^{++} \leqslant \alpha^+$.

For infinite cardinals $\alpha \leqslant \kappa$, proceed by induction on α.

If α is less than or equal to the least inaccessible cardinal μ_0, then $2^\alpha = \alpha^+$ in $V[G]$, since $\mathbb{P}^{\mu_0}$ is trivial and $\mathbb{P}^\kappa_{\mu_0}$ is $\leqslant\alpha$-closed.

If α lies in the interval (μ_0, κ) and is not inaccessible and does not lie in any interval (λ_μ, μ), for $\mu \leqslant \kappa$, then there are two cases to consider, namely, that α is singular and that there exists a largest inaccessible less than α.

If α is singular, then by induction α is a strong limit cardinal in $V[G]$. It follows by the covering claim of the previous lemma that $2^\alpha = \alpha^+$ in $V[G]$.

And if there exists a largest inaccessible μ less than α, using that $|\mathbb{P}^\mu| \leqslant \mu^{++} \leqslant \alpha^+$ and that $\mathbb{P}^\kappa_\mu$ is $\leqslant\alpha$-closed, it follows that $2^\alpha = \alpha^+$ in $V[G]$.

Now suppose that α lies in an interval (λ_μ, μ), for some inaccessible $\mu < \kappa$. It suffices to see that $\mathbb{P}^\mu$ adds μ^{++} distinct functions from λ_μ^+ into $\mathcal{P}(\lambda_\mu)$ because $\left(2^{\lambda_\mu}\right)^{\lambda_\mu^+} = 2^{\lambda_\mu^+} \leqslant 2^\alpha$. (Conversely, $2^\alpha \leqslant \mu^{++}$ because $|\mathbb{P}^\mu| \leqslant \mu^{++}$ and $\mathbb{P}^\kappa_\mu$ is $\leqslant\alpha$-closed.) Begin by noting that if $p \in \mathbb{P}^\mu$, then $\left|\bigcup_{\delta \in \mathrm{sp}(p)} \mathrm{dom}(p_\delta)\right| \leqslant \lambda_\mu$. Consequently, given any two disjoint λ_μ^+-sequences $\vec{t}$ and $\vec{t}\,'$ from T_μ, there exists a dense collection of p such that p_γ assigns different elements of T_γ to the i^{th} element of $\vec{t}$ and the i^{th} element of $\vec{t}\,'$, for some $\gamma < \lambda_\mu$ and some $i < \lambda_\mu^+$. Thus if G is generic over $\mathbb{P}^\mu$, then the following two sequences are distinct:

$$\left\langle \left\{ p_\gamma(\vec{t}_i) : \gamma < \lambda_\mu \text{ is inaccessible} \right\} : i < \lambda_\mu^+ \right\rangle$$
$$\text{and}$$
$$\left\langle \left\{ p_\gamma(\vec{t}_i\,') : \gamma < \lambda_\mu \text{ is inaccessible} \right\} : i < \lambda_\mu^+ \right\rangle.$$

Finally, if $\alpha \leqslant \kappa$ is inaccessible, it suffices to see that forcing with $\mathbb{P}^\alpha$ adds α^{++} many functions from α into $\mathcal{P}(\alpha)$. For this, note that if $p \in \mathbb{P}^\alpha$, then $\left|\bigcup_{\delta \in \mathrm{sp}(p)} \mathrm{dom}(p_\delta)\right| < \alpha$ and proceed as in the previous case. □

Proof of Silver's theorem

We are now prepared to finish the proof of Silver's theorem. Suppose that κ is λ-supercompact, where $\lambda = \kappa^{++}$. Suppose that $j\colon V \to M$ is elementary, where κ is the critical point of j and ${}^\lambda M \subseteq M$. Set $\mathbb{P} = \mathbb{P}^\kappa$.

Since $\mathbb{P} \subseteq H_\lambda$ and $|\mathbb{P}| = \lambda$, we have that $\mathbb{P} \in M$. Furthermore, $j(\mathbb{P})^\kappa = \mathbb{P}$. Set $\mathbb{Q} = \left(j(\mathbb{P})^{j(\kappa)}_\kappa\right)^M$. Then "$\mathbb{P} \times \mathbb{Q}$ is equivalent to $j(\mathbb{P})$" holds in M, hence in V. Let $e\colon \left\{ p \in j(\mathbb{P}) : \kappa \in \mathrm{sp}(p) \right\} \to \mathbb{P} \times \mathbb{Q}$ be as in the proof of the Factor Lemma. Because "$\mathbb{Q}$ is $\leqslant\lambda$-closed" holds in M and ${}^\lambda M \subseteq M$, we have that $\mathbb{Q}$ is, in fact, $\leqslant\lambda$-closed in V.

Define the master condition $\hat{q} \in \mathbb{Q}$ as follows: Set $\mathrm{sp}(\hat{q}) = \{\kappa\}$ and declare that $\hat{q}\big(j(t)\big) = t$, for each $t \in T_\kappa$. Note that $\hat{q} \in \mathbb{Q}$, using that $|j \restriction T_\kappa| = \lambda$, hence $j \restriction T_\kappa \in M$. Note also that if $p \in \mathbb{P}$, then $\mathrm{sp}\big(j(p)\big) = \mathrm{sp}(p)$ and $j(p)_\alpha = p_\alpha \circ \hat{q}$, for all $\alpha \in \mathrm{sp}(p)$. That is, in the notation of the Factor Lemma, $e\big(j(p)\big) = (p, \hat{q})$.

Suppose that G is $\mathbb{P}$ generic over V. Choose H to be $\mathbb{Q}$ generic over $V[G]$ with $\hat{q} \in H$. Set

$$K = \left\{ r \in j(\mathbb{P}) : r \geqslant r', \text{ for some } r' \text{ such that } u \in \mathrm{sp}(r') \text{ and } e(r') \in G \times H \right\}.$$

Then K is $j(\mathbb{P})$ generic over V, hence over M. Now

$$p \in G \quad \text{iff} \quad (p, \hat{q}) \in G \times H \quad \text{iff} \quad e\big(j(p)\big) \in G \times H \quad \text{iff} \quad j(p) \in K.$$

It follows that j extends to an elementary $\hat{\jmath}\colon V[G] \to M[K]$. (Set $\hat{\jmath}\big(\mathring{x}^{V[G]}\big) = j(\mathring{x})^{V[G]}$ and note that

$$\begin{aligned} V[G] \vDash \varphi(\mathring{x}) \quad &\Rightarrow \quad V \vDash p \Vdash_{\mathbb{P}} \varphi(\mathring{x}), \text{ for some } p \in G \\ &\Rightarrow \quad M \vDash j(p) \Vdash_{j(\mathbb{P})} \varphi\big(j(\mathring{x})\big), \text{ for some } p \in G \\ &\Rightarrow \quad M[K] \vDash \varphi\big(j(\mathring{x})\big).\ ) \end{aligned}$$

Working in $V[K]$, define an ultrafilter $\mathcal{U}$ on κ by

$$X \in \mathcal{U} \qquad \text{iff} \qquad \kappa \in \hat{\jmath}(X).$$

Then $\mathcal{U}$ is a $<\kappa$-complete normal non-principal ultrafilter. But $|\mathcal{U}| = \kappa^{++} = \lambda$ and $\mathbb{Q}$ is $\leqslant\lambda$-closed, so $\mathcal{U} \in V[G]$.

Hence κ is measurable and $2^\kappa = \kappa^{++}$ in $V[G]$. $\square$