INVISIBLE GENERICITY AND $0^\#$

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Abstract. $0^\#$ can be invisibly class generic.

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1. Introduction

Roughly, the main theorem of this paper is that some instances of any type of non-constructible object are class generic over L , in one sense. Since small large cardinal properties are inherited in L , let us begin by considering the sense in which $0^\#$ can be generic.

It is well known that $0^\#$ is not set generic over L , or, indeed, over any inner model M such that $0^\# \notin M$. This is because, if $0^\# \notin M$, then unboundedly many M -cardinals are collapsed in $L[0^\#]$, a trick no set forcing can perform.

It is also known that there can be no L -definable class forcing $\mathbb{P}$ such that $0^\# \in L[G]$ and $\langle L[G]; G \rangle \models \text{ZFC}$, for some L -definably generic $G \subseteq \mathbb{P}$. This is because it is known that in such a case the forcing relation on a cone of conditions rooted in G must be L -definable because $\langle L[G]; L, \mathbb{P}, G \rangle \models \text{ZFC}$; but then

$$\exists p \exists \iota \left(p \Vdash_{\mathbb{P}} \text{“}\iota \text{ is an indiscernible” and } L_\iota \models \varphi \right)$$

defines L -truth over L .

Further, it is known that there is no L -amenable class forcing property $\mathbb{P}$ such that $0^\# \in L[G]$ and $\langle L[G]; \mathbb{P}, G \rangle \models \text{ZFC}$, for some $\langle L; \mathbb{P} \rangle$ -definably generic $G \subseteq \mathbb{P}$. In such a case, again, the forcing relation on a cone of conditions rooted in G must be definable over $\langle L; \mathbb{P} \rangle$. Since $\mathbb{P}$ is L -amenable (and compatible with $0^\#$), the class $\mathbb{P}$ is definable over $L[0^\#]$, and, in fact, the members of a tail of the indiscernibles are indiscernible in $\langle L; \mathbb{P} \rangle$. But then, again, $\langle L; \mathbb{P} \rangle$ -truth is definable over $\langle L; \mathbb{P} \rangle$.

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THEOREM 1. *Suppose that $0^\#$ exists and that there exists an inaccessible cardinal in $L[0^\#]$. Then there exists a countable ordinal κ and an L_κ -amenable partial ordering $\mathbb{P}$ such that $\langle L_\kappa; \mathbb{P} \rangle \models \text{ZFC}$ and such that if $G \subseteq \mathbb{P}$ is $\langle L_\kappa; \mathbb{P} \rangle$ -definably generic, then $L_\kappa[G] \models \text{ZFC} + “0^\# \text{ exists}.”$*

This confirms a conjecture of Sy Friedman.

The subset of ω satisfying “ x is $0^\#$ ” in $L_\kappa[G]$ may or may not be the “real $0^\#$,” that is, the set assumed to exist in the hypothesis of Theorem 1.

Note the omission of the predicates $\mathbb{P}$ and G in the conclusion of the theorem. It is in this sense that $0^\#$ can be “invisibly generic.”

As suggested at the outset, the theorem proved in §2 is substantially more general than Theorem 1. For example, any large cardinal incompatible with $V=L$ can be invisibly generic in the same sense.

THEOREM 2. *Suppose that $T \supseteq \text{ZFC}$ is a first-order theory in the language of set theory and that T has a countable standard transitive model $\mathbb{V}$. Let κ be the least ordinal not in $\mathbb{V}$ and suppose that $T \in L_\kappa$ and that κ is definably regular in L_δ , where δ is the least admissible ordinal greater than κ . Then there exists an L_κ -amenable forcing property $\mathbb{P}_T$ such that*

- (1) $\langle L_\kappa; \mathbb{P}_T \rangle$ is a model of ZFC in the language $\{\in, \underline{\mathbb{P}}\}$ that includes a predicate symbol for $\mathbb{P}_T$; and
- (2) if G is $\mathbb{P}_T$ generic for $\langle L_\kappa; \mathbb{P}_T \rangle$ -definable predense classes, then $L_\kappa[G] \models T$.

In addition to Theorem 2, this paper considers four related matters. The first two, which appear in §3, are remarks on the proof of Theorem 2. One is that if $T \not\models V=L$, then T has 2^ω many non-isomorphic (in the language of set theory) $\mathbb{P}_T$ -generic models. The other is that the forcing relation for $\mathbb{P}_T$ is definable.

The second two, which appear in §4, regard the hypothesis in Theorem 2 that κ is definably regular in L_δ . Proposition 4.1 states that κ must be regular in L_δ . Proposition 4.2 examines the extent to which κ must lose its regularity in the presence of a generic $0^\#$. Very roughly, it holds that no “good” $0^\#$ can be “very generic.”

Finally, in §5 some open problems are discussed.

The following two subsections provide some definitions and references.

1.1. Class genericity

As is customary, we systematically confuse a standard structure $\langle M; \in \restriction M^2 \rangle$ with its universe M . Similarly, we confound partial orderings and their fields of conditions.

Suppose that M is a transitive set or class. When we say that $\langle M; \dots \rangle \models \text{ZFC}$ we mean that the standard structure M , augmented by the predicates, functions, or constants $\dots$, satisfies ZFC as formulated in a language with symbols for these predicates, functions, and constants. Specifically, ZFC in such a language includes all instances of collection and separation in the augmented language. Because of the importance of auxiliary predicates, rather than relying on this notation, the relevant language often will be indicated.

Although, as the statements of the theorems above indicate, we shall be working largely with countable standard models of ZFC, in order to make our terminology

widely applicable, let us frame as much of it as possible in a second-order set theory (class theory) extending ZFC and sanctioning quantification over classes.

Suppose that $\mathbb{P}$ is a partial ordering, perhaps a proper class. If $G \subseteq \mathbb{P}$, then G is a *filter* if G consists of pairwise compatible conditions and G is closed upwards: $p \in G$ whenever there exists a $q \in G$ such that $q \leq p$.

Suppose that M is an inner model, that is, a transitive class containing all of the ordinals and satisfying each axiom of ZFC. Define the class $M^\mathbb{P}$ of *Shoenfield terms* by recursion:

$$\dot{a} \in M^\mathbb{P} \quad \text{iff} \quad \dot{a} \subseteq M^\mathbb{P} \times \mathbb{P} \text{ and } \dot{a} \text{ is a set of } M.$$

For Shoenfield terms $\dot{a}$, set

$$\begin{aligned} \text{dom}(\dot{a}) &= \left\{ \dot{b} : (\dot{b}, p) \in \dot{a}, \text{ for some } p \right\} \quad \text{and} \\ \dot{a}(\dot{b}) &= \left\{ p \in \mathbb{P} : (\dot{b}, p) \in \dot{a} \right\}, \quad \text{for } \dot{b} \in \text{dom}(\dot{a}). \end{aligned}$$

If G is a filter on $\mathbb{P}$, define the *value* of the Shoenfield term $\dot{a}$ by

$$\begin{aligned} \dot{a}^G &= \left\{ \dot{b}^G : \dot{b} \in \text{dom}(\dot{a}) \text{ and } \dot{a}(\dot{b}) \cap G \neq \emptyset \right\}; \quad \text{and set} \\ M[G] &= \left\{ \dot{a}^G : \dot{a} \in M^\mathbb{P} \right\}. \end{aligned}$$

Then $M[G]$ is a transitive class.

Suppose that x is a set. Then x is *set generic* over M if there exists a partial ordering $\mathbb{P} \in M$ and a filter G such that $x \in M[G] \setminus M$ and $G \cap D \neq \emptyset$, for all predense $D \subseteq \mathbb{P}$ that are elements of M .

Say that a set x is *M -amenably generic* when there exists an (M -amenable) partial ordering $\mathbb{P}$ and a filter G such that

- (1) $x \in M[G] \setminus M$;
- (2) $\langle M; \mathbb{P} \rangle$ satisfies ZFC in a language with predicate symbols for $\mathbb{P}$;
- (3) G is $\langle M; \mathbb{P} \rangle$ -*definably generic*, that is, $G \cap D \neq \emptyset$, for all $D \subseteq \mathbb{P}$ that are predense and definable over $\langle M; \mathbb{P} \rangle$; and
- (4) $\langle M[G]; M, \mathbb{P}, G \rangle$ satisfies ZFC in a language with predicate symbols for M , $\mathbb{P}$, and G .

A set x is *weakly generic* when x is M -amenably generic over some inner model M .

Say that a set x is *invisibly generic* if, for some inner model M , it satisfies the definition of M -amenable genericity with the following weakened clause (4):

- (4') $M[G]$ satisfies ZFC in the language of set theory.

So x 's genericity is perhaps "invisible" in $M[G]$.

This last notion is dicey in class theory, since ZFC fails in $\langle M[G]; M, \mathbb{P}, G \rangle$ unless x is in fact weakly generic. Consequently, we shall work with set models when dealing with invisible genericity.

Let us recapitulate in this terminology $0^\#$'s status: It is known that $0^\#$ is neither set generic nor L -amenably generic. Theorem 1 is that it can be invisibly generic. It remains open whether $0^\#$ can be weakly generic.

1.2. $\mathbb{H}(\mathbb{A})$

In this section we introduce some notation and state and give references for several standard facts from admissible set theory that figure in later sections.

If X is a transitive set, let

$$o(X) = X \cap \text{OR}.$$

Then $o(X)$ is the least ordinal that does not lie in X .

Suppose that A is a transitive set and that $R_1, \dots, R_n$ are amenable relations on A . Set $\mathbb{A} = \langle A; R_1, \dots, R_n \rangle$ and define $L_\alpha(\mathbb{A})$ by

$$\begin{aligned} L_0(\mathbb{A}) &= \emptyset; \\ L_{\alpha+1}(\mathbb{A}) &= \bigcup_{a \in \mathbb{A}} \text{cl}\left(L_\alpha(\mathbb{A}) \cup \{L_\alpha(\mathbb{A})\} \cup \{L_\alpha(\mathbb{A}) \cap a\}\right) \cap \mathcal{P}(L_\alpha(\mathbb{A})); \text{ and} \\ L_\alpha(\mathbb{A}) &= \bigcup_{\gamma < \alpha} L_\gamma(\mathbb{A}), \quad \text{if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

Here $\text{cl}(X)$ is the closure of X under the usual Gödel operations, together with the Gödel operations for $R_1, \dots, R_n$, namely $\mathcal{F}_{R_i}(x) = x \cap R_i$.

Say that $\mathbb{A}$ is *power admissible* when it satisfies KP in a language with a function symbol $\mathcal{P}$ for the power set operation; that is, $\mathbb{A}$ satisfies KP, including instances of Σ_1 -collection and Σ_0 -separation involving $R_1, \dots, R_n$ and $\mathcal{P}$, as well as satisfies the axiom

$$\forall x \forall y (y \in \mathcal{P}(x) \leftrightarrow y \subseteq x).$$

Models of ZFC have natural expansions to power admissible sets.

If $\mathbb{A}$ is power admissible, then $\mathbb{A}$ can be internally resolved into ranks $\mathbb{V}_\alpha$, for $\alpha < o(\mathbb{A})$:

$$\begin{aligned} \mathbb{V}_0 &= \emptyset; \\ \mathbb{V}_{\alpha+1} &= \mathcal{P}^{\mathbb{A}}(\mathbb{V}_\alpha); \text{ and} \\ \mathbb{V}_\alpha &= \bigcup_{\gamma < \alpha} \mathbb{V}_\gamma, \quad \text{if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

In other words $\mathbb{V}_\alpha = V_\alpha^{\mathbb{A}}$. [Our convention is to use blackboard bold for external names of standard transitive structures (and partial orderings). The internal/external distinction is admittedly obscure, since our point of view is continually shifting.]

Note that $L_\alpha(\mathbb{A}) = \mathbb{V}_\alpha$, for $\alpha < o(\mathbb{A})$. Thus $\mathbb{A} = L_{o(\mathbb{A})}(\mathbb{A})$ and $\mathbb{A} \in L_{o(\mathbb{A})+\omega}(\mathbb{A})$.

Let $\alpha > o(\mathbb{A})$ be least such that $L_\alpha(\mathbb{A})$ is admissible and set

$$\mathbb{H}(\mathbb{A}) = L_\alpha(\mathbb{A}).$$

Then $\mathbb{H}(\mathbb{A})$ is the smallest admissible set with $\mathbb{A} \in \mathbb{H}(\mathbb{A})$. For example, if $\mathbb{A} = L_\kappa$, then $\mathbb{H}(\mathbb{A}) = L_\delta$, where δ is the least admissible ordinal greater than κ .

If $X \subseteq \mathbb{A}$ is first-order definable over $\mathbb{A}$, then $X \in \mathbb{H}(\mathbb{A})$. (In fact, if $\mathbb{A}$ is countable, then this holds for X that are Δ_1^1 definable over $\mathbb{A}$, but we shall not need this fact.)

Suppose that $\triangleleft$ is a binary relation on some set A . Define the well-founded part of $\triangleleft$ by

$$\text{wf}(\triangleleft) = \left\{ b \in A : \langle \text{pred}_{\triangleleft}(b); \triangleleft \rangle \text{ is well-founded} \right\},$$

where $\text{pred}_{\triangleleft}(b) = \{ a \in A : a \triangleleft b \}$. Define the rank function $\rho_{\triangleleft}$ on $\text{wf}(\triangleleft)$ as usual:

$$\rho_{\triangleleft}(b) = \sup \left\{ \rho_{\triangleleft}(a) + 1 : a \triangleleft b \right\}.$$

Then set $\rho(\triangleleft) = \sup \{ \rho_{\triangleleft}(b) + 1 : b \in \text{wf}(\triangleleft) \}$. The following can be found in [B, §V.3]:

LEMMA. Suppose $\triangleleft \subseteq \mathbb{A}^2$ and $\triangleleft \in \mathbb{H}(\mathbb{A})$. Then

- (a) $\rho(\triangleleft) \leq o(\mathbb{H}(\mathbb{A}))$;
- (b) if $\text{wf}(\triangleleft) \in \mathbb{H}(\mathbb{A})$, then $\rho(\triangleleft) \in \mathbb{H}(\mathbb{A})$; and
- (c) if $\text{wf}(\triangleleft) \notin \mathbb{H}(\mathbb{A})$, then $\rho(\triangleleft) = o(\mathbb{H}(\mathbb{A}))$.

Because $\mathbb{H}(\mathbb{A})$ is the smallest admissible set above $\mathbb{A}$, the admissible set $\mathbb{H}(\mathbb{A})$ is *projectible* into $\mathbb{A}$. A definition of this notion, together with a proof of the following fact can be found in [B, §V.5]:

$$o(\mathbb{H}(\mathbb{A})) = \left\{ \rho(\triangleleft) : \triangleleft \subseteq \mathbb{A}^2 \text{ is well-founded and } \triangleleft \in \mathbb{H}(\mathbb{A}) \right\}$$

2. Almost anything might be invisibly generic

The main theorem of this paper is

THEOREM 2. Suppose that $T \supseteq \text{ZFC}$ is a first-order theory in the language of set theory and that T has a countable standard transitive model $\mathbb{V}$ with $o(\mathbb{V}) = \kappa$. Suppose as well that $T \in L_\kappa$ and that κ is definably regular in $\mathbb{H}(L_\kappa)$. Then there exists an L_κ -amenable forcing property $\mathbb{P}_T$ such that

- (1) $\langle L_\kappa; \mathbb{P}_T \rangle$ is a model of ZFC in the language $\{\in, \underline{\mathbb{P}}\}$ that includes a predicate symbol for $\mathbb{P}_T$; and
- (2) if G is $\mathbb{P}_T$ generic for $\langle L_\kappa; \mathbb{P}_T \rangle$ -definable predense classes, then $L_\kappa[G] \models T$.

Again, let us remind the reader that $L_\kappa[G]$ is the standard transitive structure consisting of the values $\dot{a}^G$ of the Shoenfield terms in the $(L_\kappa\text{-amenable})$ forcing language $L_\kappa^{\mathbb{P}_T}$.

We say that κ is *definably regular* in $\mathbb{H}(L_\kappa)$ when there is no $\mathbb{H}(L_\kappa)$ -definable cofinal function $f: \lambda \rightarrow \kappa$, for some $\lambda < \kappa$; or equivalently, when κ remains regular in $L_{\delta+1}$, where $L_\delta = \mathbb{H}(L_\kappa)$. It follows from this hypothesis and the hypothesis that κ is $o(\mathbb{V})$, where $\mathbb{V}$ is a model of ZFC, that κ is an inaccessible cardinal in $\mathbb{H}(L_\kappa)$.

Note also that, for all n , we have that κ is the Σ_n -projectum of δ , where $L_\delta = \mathbb{H}(L_\kappa)$. This observation will be used in the proof of Remark 3.1.

The following corollary is immediate from Theorem 2. Theorem 1, that $0^\#$ can be invisibly generic, is a special case of it.

COROLLARY 2.1. Suppose that φ is a formula in the language of set theory, that the theory $\text{ZFC} + \exists x \varphi(x) \vdash V \neq L$, and that $V_\lambda \models \text{ZFC} + \exists x \varphi(x)$, for some inaccessible λ . Then there exists a countable ordinal κ and an L_κ -amenable forcing property $\mathbb{P}$ such that

- (1) $\langle L_\kappa; \mathbb{P} \rangle \models \text{ZFC} + \neg \exists x \varphi(x)$ and
- (2) if G is $\mathbb{P}$ generic for $\langle L_\kappa; \mathbb{P} \rangle$ -definable predense classes, then $L_\kappa[G] \models \text{ZFC} + \exists x \varphi(x)$.

That is, $\varphi(x)$ is solvable in an invisibly generic extension of L_κ .

The forcing property used to prove Theorem 2 will construct a generic Henkin model of the theory T . The work goes into arranging that exactly the elements of this model are denoted by Shoenfield terms. So there are two sorts of terms that we

shall need, namely, the logical terms needed to construct a Henkin model, and the Shoenfield terms whose denotations constitute the generic extension itself. Our first step is to introduce a one-place function symbol $\underline{F}$ to provide an adequate supply of logical terms.

Let $\underline{F}$ be a one-place function symbol and let $(\dagger)$ be a formula of the language $\{\in, \underline{F}\}$ formalizing the following:

$$(\dagger) \quad \begin{aligned} & \forall x \exists \alpha (\alpha \in \text{OR} \wedge \underline{F}(\alpha) = x) \wedge \\ & \forall \alpha \forall \beta (\alpha, \beta \in \text{OR} \wedge \underline{F}(\beta) \in \underline{F}(\alpha) \rightarrow \beta < \alpha) \end{aligned}$$

Of course, any model of $\text{ZFC} + V=L[0^\#]$ has an expansion by definitions to a model of $\text{ZFC} + V=L[0^\#] + (\dagger)$ given by interpreting $\underline{F}$ as F , where

$$F(x) = \begin{cases} x^{\text{th}} \text{ element of } L[0^\#] \text{ under } <_{L[0^\#]}, & \text{if } x \text{ is an ordinal;} \\ \emptyset, & \text{otherwise.} \end{cases}$$

More generally we have

LEMMA 2.2. *Suppose that $T \supseteq \text{ZFC}$ is a theory in the language of set theory and that $\mathbb{V}$ is a countable standard transitive model of T . Then there exists a $\mathbb{V}$ -amenable function F such that $\langle \mathbb{V}; F \rangle$ is a model of $T + (\dagger) + \text{ZFC}$, where ZFC is formulated in the language $\{\in, \underline{F}\}$.*

PROOF: Working in $\mathbb{V}$ define a class forcing property $\mathbb{F}$ by

$$\begin{aligned} f \in \mathbb{F} \quad & \text{iff} \quad f \text{ is a one-to-one function with } \text{dom}(f) \in \text{OR} \text{ and} \\ & \text{rng}(f) = V_\beta, \text{ for some } \beta, \text{ and satisfying that } \beta < \alpha \\ & \text{whenever } f(\beta) \in f(\alpha). \\ f' \leq f \quad & \text{iff} \quad f' \supseteq f \end{aligned}$$

Then $\mathbb{F}$ is $<\infty$ -closed. Using the axiom of choice in $\mathbb{V}$, an induction on β shows that any V_β can be absorbed into the range of a condition extending a given condition. If G is $\mathbb{F}$ generic for $\mathbb{V}$ -definable predense classes, then $F = \bigcup G \cup \{(x, \emptyset) : x \notin \text{OR}\}$ is as required. $\square$

Note that $(\dagger)$ does not require that F restricted to the ordinals is one-to-one, even though both examples given here ($<_{L[0^\#]}$ -enumeration and the previous lemma) have this additional property. That F is not required to be one-to-one is used in the proof of Lemma 2.4 (to see that $\{\theta_\kappa, \underline{F}(\beta) \in \underline{F}(\alpha)\}$ is consistent).

Theorem 2 will be proved by establishing three lemmas. The first of these lemmas states that a standard transitive Henkin model $\mathbb{V}_G$ of T can be defined from any $\langle L_\kappa, \mathbb{P}_T \rangle$ -definably generic G . The second shows that $\mathbb{V}_G \subseteq L_\kappa[G]$, and the third that $L_\kappa[G] \subseteq \mathbb{V}_G$.

The forcing $\mathbb{P}_T$ generically builds $\mathbb{V}_G$ as a Henkin model of T in an infinitary language. There are several ways such a forcing could be defined. As suggested above, our goal is to define $\mathbb{P}_T$ in such a way that there are enough terms in $L_\kappa^{\mathbb{P}_T}$ to make the second lemma true, but not so many that the third fails.

Conditions will be pairs $(r, \vec{s})$, where r is a finite piece of the finitary diagram of $\mathbb{V}_G$, and $\vec{s}$ is a finite sequence of approximations to the infinitary elementary diagram of $\mathbb{V}_G$. The idea is that the class of r 's added by G is accessible to $L_\kappa[G]$ and so provides an adequate supply of Shoenfield terms, but the $\vec{s}$'s fit together to form an amenable ω -sequence that cannot be defined over $L_\kappa[G]$. The $\vec{s}$'s are used to hide the scratch work needed to construct $\mathbb{V}_G$.

Let us begin by setting up some notation and defining $\mathbb{P}_T$. Fix a first order theory $T \supseteq \text{ZFC}$ in the language of set theory. Fix a countable ordinal κ such that $T \in L_\kappa$ and κ is definably regular in $\mathbb{H}(L_\kappa)$ and T has a standard transitive model $\mathbb{V}$ with $o(\mathbb{V}) = \kappa$. Working in $\mathbb{H}(L_\kappa)$, define $X \subseteq \kappa + 1$ by

$$\gamma \in X \quad \text{iff} \quad \gamma \leq \kappa \quad \text{and} \quad \left(\Sigma_2\text{-Skolem Hull}_{\mathbb{H}(L_\kappa)}(\gamma \cup \{\kappa\}) \right) \cap \kappa = \gamma.$$

Then $\kappa \in X$ and, because κ is definably regular in $\mathbb{H}(L_\kappa)$, the set $X \cap \kappa$ is unbounded in κ . Note that if $\gamma \in X$, then

$$\Sigma_2\text{-Skolem Hull}_{\mathbb{H}(L_\kappa)}(\gamma \cup \{\kappa\}) \cong \mathbb{H}(L_\gamma).$$

It is for this that we take Σ_2 -Skolem hulls, even though each element of $\mathbb{H}(L_\kappa)$ is Σ_1 -definable from parameters in $\kappa \cup \{\kappa\}$.

It follows that there exists a commutative system of Σ_2 -elementary embeddings $\{j_{\bar{\gamma}\gamma} : \bar{\gamma} \leq \gamma \text{ in } X\}$ such that $j_{\bar{\gamma}\gamma} : \mathbb{H}(L_{\bar{\gamma}}) \rightarrow \mathbb{H}(L_\gamma)$; $j_{\bar{\gamma}\gamma} \upharpoonright \bar{\gamma} = \text{id} \upharpoonright \bar{\gamma}$; $j_{\bar{\gamma}\gamma}(\bar{\gamma}) = \gamma$; and $\mathbb{H}(L_\kappa) = \bigcup_{\gamma \in X \cap \kappa} \text{rng}(j_{\bar{\gamma}\gamma})$.

Let $\mathcal{L}$ be the language with $\approx$ (equality) having the following symbols:

- a two-place relation symbol $\in$;
- a one-place function symbol $\underline{F}$; and
- constant symbols $\underline{\alpha}$, for $\alpha < \kappa$.

We shall make use of admissible fragments $\mathcal{L}_{\mathbb{H}(L_\gamma)}$, for $\gamma \in X$. (For example, see [B, Chapter III] regarding the syntax and semantics of these languages.) As is customary, we identify formulas with sets that code them. Let us assume that logical symbols, as well as the symbols $\in$ and $\underline{F}$, are coded by hereditarily finite sets, and that the symbol $\underline{\alpha}$ is coded by α itself.

The non-logical symbols of the language $\mathcal{L}_{\mathbb{H}(L_\gamma)}$ are $\in$, $\underline{F}$, and $\underline{\alpha}$, for $\alpha < \gamma$. The formulas of $\mathcal{L}_{\mathbb{H}(L_\gamma)}$ are those $\mathcal{L}_{\infty\omega}$ formulas formed using these symbols and lying in the admissible set $\mathbb{H}(L_\gamma)$ (and having only finitely many free variables).

A *basic sentence* of $\mathcal{L}_{\mathbb{H}(L_\gamma)}$ is an atomic or negated atomic sentence.

For $\gamma \in X$, let θ_γ be an $\mathcal{L}_{\mathbb{H}(L_\gamma)}$ sentence that results from conjoining formalizations of the following:

- T ;
- ZFC in the language $\{\in, \underline{F}\} \cup \{\underline{\alpha} : \alpha < \gamma\}$;
- $(\dagger)$;
- $\forall x (x \in \underline{\beta} \leftrightarrow \bigvee_{\delta < \beta} x \approx \underline{\delta})$, for each $\beta < \gamma$; and
- $\forall x (x \in \text{OR} \leftrightarrow \bigvee_{\delta < \gamma} x \approx \underline{\delta})$.

By hypothesis we have that θ_κ is satisfiable, hence $\{\theta_\kappa\}$ is consistent in $\mathcal{L}_{\mathbb{H}(L_\kappa)}$.

For $\gamma \in X$, declare that

$$s \in \mathbb{S}_\gamma \quad \text{iff} \quad s \text{ is a finite set of } \mathcal{L}_{\mathbb{H}(L_\gamma)} \text{ sentences and} \\ s \cup \{\theta_\gamma\} \text{ is consistent in } \mathcal{L}_{\mathbb{H}(L_\gamma)}.$$

That is, a finite set s of $\mathcal{L}_{\mathbb{H}(L_\gamma)}$ sentences is in $\mathbb{S}_\gamma$ when there is no infinitary proof of the sentence $(\theta_\gamma \wedge \bigwedge s \rightarrow \underline{0} \approx \underline{1})$. By Barwise Completeness, $s \in \mathbb{S}_\gamma$ iff $\mathbb{H}(L_\gamma) \models "s \in \mathbb{S}_\gamma."$ Note also that

$$s \in \mathbb{S}_{\bar{\gamma}} \quad \text{iff} \quad j_{\bar{\gamma}\gamma}(s) \in \mathbb{S}_\gamma,$$

whenever $\bar{\gamma} \leq \gamma$ in X .

Declare that $(r, \vec{s}) \in \mathbb{P}$ iff $(r, \vec{s})$ meets the following four requirements:

- (1) r is a finite set of basic $\mathcal{L}_{\mathbb{H}(L_\kappa)}$ sentences;
- (2) $\vec{s} = \langle (s_i, \gamma_i) : i < n \rangle$, for some $n \in \omega$, where $\gamma_i \in X \cap \kappa$ and $s_i \in \mathbb{S}_{\gamma_i}$; and
- (3) if $i < k < n$, then $\gamma_i \leq \gamma_k$ and $s_k \supseteq j_{\gamma_i \gamma_k}(s_i)$.

Let $\|r\|$ be the least $\gamma \in X$ such that $r \subseteq L_\gamma$. Finally, require that

- (4) $r \cup j_{\gamma_{n-1}\gamma}(s_{n-1}) \in \mathbb{S}_\gamma$, if $n = |\vec{s}| > 0$ and $\gamma \in X \setminus \max(\gamma_{n-1}, \|r\|)$; if $|\vec{s}| = 0$, simply require that $r \in \mathbb{S}_{\|r\|}$.

Note that in clause (4) whether $r \cup j_{\gamma_{n-1}\gamma}(s_{n-1}) \in \mathbb{S}_\gamma$ does not depend on the choice of $\gamma \in X \setminus \max(\gamma_{n-1}, \|r\|)$.

Order $\mathbb{P}$ by declaring that

$$(r', \vec{s}') \leq (r, \vec{s}) \quad \text{iff} \quad r' \supseteq r \text{ and } \vec{s} \text{ is an initial segment of } \vec{s}'.$$

Note that $\mathbb{P}$ is definable over $\mathbb{H}(L_\kappa)$ and so $\mathbb{P}$ is L_κ -amenable and $\langle L_\kappa; \mathbb{P} \rangle \models \text{ZFC}$ in a language with a predicate symbol for $\mathbb{P}$.

LEMMA 2.3. *Suppose that G is $\mathbb{P}$ generic for $\langle L_\kappa; \mathbb{P} \rangle$ -definable predense classes. Then there exists a canonical countable standard transitive model $\langle \mathbb{V}_G; F_G, \alpha \rangle_{\alpha < \kappa}$ of θ_κ such that*

$$\langle \mathbb{V}_G; F_G, \alpha \rangle_{\alpha < \kappa} \models \bigwedge r \quad \text{iff} \quad (r, \emptyset) \in G,$$

for all finite sets r of basic $\mathcal{L}_{\mathbb{H}(L_\kappa)}$ sentences.

PROOF: Fix $\gamma \in X$. First, let us establish, roughly, that $\mathbb{S}_\gamma$ is a “consistency property” in the sense of [B].

$$(D0) \quad \emptyset \in \mathbb{S}_\gamma$$

To see this, note first that since θ_κ is consistent and $j_{\gamma\kappa}(\theta_\gamma) = \theta_\kappa$, we have that $\mathbb{H}(L_\gamma) \models " \theta_\gamma \text{ is consistent.}"$ By Barwise Completeness it follows that $\{\theta_\gamma\} \cup \emptyset$ is, in fact, consistent.

Suppose that $s \in \mathbb{S}_\gamma$. The syntax of the infinitary language $\mathcal{L}_{\mathbb{H}(L_\gamma)}$ insures the following:

- (D1) If $\varphi \in s$, then $(\neg\varphi) \notin s$.
- (D2) If $(\neg\varphi) \in s$, then $s \cup \{\sim\varphi\} \in \mathbb{S}_\gamma$.

Here $\sim \varphi$ denotes the result of pushing the initial negation in $(\neg \varphi)$ inside φ one step. (For example, $\sim(\neg \varphi) = \varphi$ and $\sim \bigvee \Phi = \bigwedge \{ \neg \varphi : \varphi \in \Phi \}$.)

- (D3) If $\bigwedge \Phi \in s$, then $s \cup \{\varphi\} \in \mathbb{S}_\gamma$, for each $\varphi \in \Phi$.
- (D4) If $\forall v \varphi \in s$, then $s \cup \{\varphi(t/v)\} \in \mathbb{S}_\gamma$, for each closed $\mathcal{L}_{\mathbb{H}(L_\gamma)}$ term t .
- (D5) If $\bigvee \Phi \in s$, then $s \cup \{\varphi\} \in \mathbb{S}_\gamma$, for some $\varphi \in \Phi$.
- (D7) If t and t' are closed $\mathcal{L}_{\mathbb{H}(L_\gamma)}$ terms, then
 - (i) $s \cup \{t \approx t'\} \in \mathbb{S}_\gamma$;
 - (ii) if $(t \approx t') \in s$, then $s \cup \{t' \approx t\} \in \mathbb{S}_\gamma$; and
 - (iii) if $\varphi(t/v), (t \approx t') \in s$, then $s \cup \{\varphi(t'/v)\} \in \mathbb{S}_\gamma$.
- (D8) If φ is a sentence of $\mathcal{L}_{\mathbb{H}(L_\gamma)}$, then either $s \cup \{\varphi\} \in \mathbb{S}_\gamma$ or else $s \cup \{(\neg \varphi)\} \in \mathbb{S}_\gamma$.

A little argument is need to see

- (D6) If $\exists v \varphi \in s$, then $s \cup \{\varphi(t/v)\} \in \mathbb{S}_\gamma$, for some closed $\mathcal{L}_{\mathbb{H}(L_\gamma)}$ term t .

Work in the real world, where $\mathbb{H}(L_\gamma)$ is countable. Since $\{\theta_\gamma\} \cup s$ is consistent, it has a model $\mathfrak{A}$. And $\mathfrak{A} \models \forall x (x \in \text{OR} \leftrightarrow \bigvee_{\delta < \gamma} x \approx \delta)$. Thus if $d \in \text{OR}^\mathfrak{A}$, then $d = \underline{\delta}^\mathfrak{A}$, for some $\delta < \gamma$. Also $\mathfrak{A} \models (\dagger)$, so $\underline{F}^\mathfrak{A}: \text{OR}^\mathfrak{A} \rightarrow V^\mathfrak{A}$ is onto. Say $\mathfrak{A} \models \varphi[a]$, where $a = \underline{F}^\mathfrak{A}(\underline{\delta}^\mathfrak{A})$. Then $\mathfrak{A} \models \varphi(\underline{F}(\delta)/v)$. Thus $\{\theta_\gamma\} \cup s \cup \{\varphi(\underline{F}(\delta)/v)\}$ is consistent in the real world, hence in $\mathbb{H}(L_\gamma)$.

Suppose now that G is $\mathbb{P}$ generic for $\langle L_\kappa; \mathbb{P} \rangle$ -definable predense classes. Set

$$\Gamma = \bigcup \left\{ j_{\gamma_{n-1}\kappa}(s_{n-1}) : (r, \vec{s}) \in G \text{ and } n = |\vec{s}| > 0 \right\}.$$

Note first that

- ($\Gamma 0$) $\theta_\kappa \in \Gamma$.

Indeed, by (D0) we have that $\mathbb{S}_\gamma \neq \emptyset$, for all $\gamma \in X \cap \kappa$. And if $s \in \mathbb{S}_\gamma$, then certainly $s \cup \{\theta_\gamma\} \in \mathbb{S}_\gamma$. Because $j_{\gamma\kappa}(\theta_\gamma) = \theta_\kappa$ and G is generic, it suffices to see that

$$\left\{ (r, \vec{s}) \in \mathbb{P} : 0 < |\vec{s}| = n \text{ and } \theta_\gamma \in s_{n-1}, \text{ where } \gamma = \gamma_{n-1} \right\}$$

is dense in $\mathbb{P}$. Suppose $(r, \vec{s}) \in \mathbb{P}$. We may assume that $n = |\vec{s}| > 0$. Set $\bar{\gamma} = \gamma_{n-1}$ and $\gamma = \min(X \setminus \max(\bar{\gamma}, \|r\|))$. Then $r \cup j_{\bar{\gamma}\gamma}(s_{n-1}) \in \mathbb{S}_\gamma$. Let $\vec{s}'$ be the sequence extending $\vec{s}$ to length $n+1$ such that $\gamma'_n = \gamma$ and $s'_n = j_{\bar{\gamma}\gamma}(s_{n-1}) \cup \{\theta_\gamma\}$. Then $(r, \vec{s}') \leq (r, \vec{s})$ and lies in the required class of conditions.

- ($\Gamma 1$) If $\varphi \in \Gamma$, then $(\neg \varphi) \notin \Gamma$.

Suppose not. Say $\varphi \in j_{\gamma\kappa}(s)$ and $(\neg \varphi) \in j_{\gamma'\kappa}(s')$, where $(r, \vec{s}), (r', \vec{s}') \in G$; $\gamma = \gamma_{|\vec{s}|-1}$ and $s = s_{|\vec{s}|-1}$; and $\gamma' = \gamma'_{|\vec{s}'|-1}$ and $s' = s_{|\vec{s}'|-1}$. Say $\gamma \leq \gamma'$. Then $\vec{s}$ is an initial segment of $\vec{s}'$ because G is a filter. Consequently, $j_{\gamma'\kappa}^{-1}(\varphi) = j_{\gamma\gamma'}(j_{\gamma\kappa}^{-1}(\varphi)) \in s'$. But also $(\neg j_{\gamma'\kappa}^{-1}(\varphi)) = j_{\gamma'\kappa}^{-1}(\neg \varphi) \in s'$, contradicting (D1).

- ($\Gamma 2$) If $(\neg \varphi) \in \Gamma$, then $\sim \varphi \in \Gamma$.

Say $(\neg\varphi) \in j_{\gamma\kappa}(s)$, where $(r, \vec{s}) \in G$ and $|\vec{s}| = n$ and $s = s_{n-1}$ and $\gamma = \gamma_{n-1}$. We may assume that $\gamma \geq \|r\|$. Then $r \cup s \in \mathbb{S}_\gamma$, so $r \cup s \cup \{\sim j_{\gamma\kappa}^{-1}(\varphi)\} \in \mathbb{S}_\gamma$ by (D2). Let $\vec{s}'$ be the sequence of length $n+1$ extending $\vec{s}$ with $\gamma_n = \gamma$ and $s_n = s \cup \{\sim j_{\gamma\kappa}^{-1}(\varphi)\}$. Then $(r, \vec{s}') \leq (r, \vec{s})$.

It follows from this observation that the $\langle L_\kappa; \mathbb{P} \rangle$ -definable class

$$\begin{aligned} & \left\{ (r', \vec{s}') \in \mathbb{P} : \sim j_{\gamma_{|\vec{s}'|-1}\kappa}^{-1}(\varphi) \in s'_{|\vec{s}'|-1} \right\} \\ &= \left\{ (r', \vec{s}') \in \mathbb{P} : j_{\gamma_{|\vec{s}'|-1}\kappa}(\sim j_{\gamma\kappa}^{-1}(\varphi)) \in s'_{|\vec{s}'|-1} \right\} \end{aligned}$$

is dense below $(r, \vec{s})$ in $\mathbb{P}$. By genericity, we have (Γ2).

Similarly, using (D3)–(D8) respectively, we have

(Γ3) If $\bigwedge \Phi \in \Gamma$, then $\varphi \in \Gamma$, for all $\varphi \in \Phi$.

(Γ4) If $\forall v \varphi \in \Gamma$, then $\varphi(t/v) \in \Gamma$, for each closed term t of $\mathcal{L}_{\mathbb{H}(L_\kappa)}$.

(Γ5) If $\bigvee \Phi \in \Gamma$, then $\varphi \in \Gamma$, for some $\varphi \in \Phi$.

(Γ6) If $\exists v \varphi \in \Gamma$, then $\varphi(t/v) \in \Gamma$, for some closed term t of $\mathcal{L}_{\mathbb{H}(L_\kappa)}$.

(Γ7) If t and t' are closed $\mathcal{L}_{\mathbb{H}(L_\kappa)}$ terms, then

(i) $(t \approx t) \in \Gamma$;

(ii) if $(t \approx t') \in \Gamma$, then $(t' \approx t) \in \Gamma$; and

(iii) if $\varphi(t/v), (t \approx t') \in \Gamma$, then $\varphi(t'/v) \in \Gamma$.

(Γ8) If φ is a sentence of $\mathcal{L}_{\mathbb{H}(L_\kappa)}$, then either $\varphi \in \Gamma$ or $(\neg\varphi) \in \Gamma$.

In addition to (D3)–(D8) and the genericity of G , propositions (Γ4), (Γ7i), and (Γ8) use that $\mathbb{H}(L_\kappa) = \bigcup_{\gamma \in X \cap \kappa} \text{rng}(j_{\gamma\kappa})$.

Finally, note that

(Γ9) if φ is a basic sentence of $\mathcal{L}_{\mathbb{H}(L_\kappa)}$, then $\varphi \in \Gamma$ iff $\varphi \in r$, for some $(r, \vec{s}) \in G$.

This observation uses that

$$\left\{ (r, \vec{s}) \in \mathbb{P} : |\vec{s}| > 0 \text{ and } r = s_{|\vec{s}|-1} \cap \{ \text{basic sentences of } \mathcal{L}_{\mathbb{H}(L_\kappa)} \} \right\}$$

is dense in $\mathbb{P}$, and that $j_{\gamma\kappa}(\varphi) = \varphi$ when φ is basic.

Next, we use Γ to construct a model of θ_κ . Begin by defining a relation $\equiv$ on closed $\mathcal{L}_{\mathbb{H}(L_\kappa)}$ terms by

$$t \equiv t' \quad \text{iff} \quad (t \approx t') \in \Gamma.$$

By (Γ7) we have that $\equiv$ is an equivalence relation. Then there exists a canonical structure $\mathfrak{A}$ in which elements of the universe are $\equiv$ -equivalence classes and

$$\mathfrak{A} \models \varphi \quad \text{iff} \quad \varphi \in \Gamma,$$

for all sentences φ of $\mathcal{L}_{\mathbb{H}(L_\kappa)}$. (This is established by induction on sentences using (Γ1)–(Γ8).) Furthermore, $\mathfrak{A} \models \theta_\kappa$ by (Γ0), so there exists a unique standard transitive model $\langle \mathbb{V}_G; F_G, \alpha \rangle_{\alpha < \kappa} \cong \mathfrak{A}$.

Finally, suppose that r is a finite set of basic $\mathcal{L}_{\mathbb{H}(L_\kappa)}$ sentences. Then

$$\begin{aligned}
\langle \mathbb{V}_G; F_G, \alpha \rangle_{\alpha < \kappa} \models \bigwedge r & \text{ iff } \mathfrak{A} \models \bigwedge r \\
& \text{ iff } \bigwedge r \in \Gamma \\
& \text{ iff } r \subseteq \Gamma \\
& \text{ iff } r \subseteq j_{\gamma_{n-1}\kappa}(s'_{n-1}), \text{ for some } (r', \vec{s}') \in G, \\
& \text{ where } |\vec{s}'| = n > 0 \\
& \text{ iff } r \subseteq r', \text{ for some } (r', \vec{s}') \in G \\
& \text{ iff } (r, \emptyset) \in G
\end{aligned}$$

The next to last equivalence uses that G is generic; the last, that G is a filter. $\square$

LEMMA 2.4. *Suppose that G is $\mathbb{P}$ generic for $\langle L_\kappa; \mathbb{P} \rangle$ -definable predense classes. Then for each $\alpha < \kappa$ there exists a Shoenfield term $\mathring{F}(\alpha) \in L_\kappa^\mathbb{P}$ such that $F_G(\alpha) = \mathring{F}(\alpha)^G$. Thus $\mathbb{V}_G \subseteq L_\kappa[G]$.*

PROOF: Set

$$\mathring{F}(\alpha) = \left\{ \left(\mathring{F}(\beta), (\{ \underline{F}(\beta) \in \underline{F}(\alpha) \}, \emptyset) \right) : \beta < \alpha \right\}.$$

Then $\mathring{F}$ is an L_κ -definable function and $\mathring{F}(\alpha)$ is a Shoenfield term in $L_\kappa^\mathbb{P}$, for each $\alpha < \kappa$. (Note that $\{ \theta_\kappa, \underline{F}(\beta) \in \underline{F}(\alpha) \}$ is consistent.)

Proceed by induction on α to see that $\mathring{F}(\alpha)^G = F_G(\alpha)$. Note that

$$\begin{aligned}
F_G(\beta) \in F_G(\alpha) & \text{ iff } \langle \mathbb{V}_G; F_G, \alpha \rangle_{\alpha < \kappa} \models \underline{F}(\beta) \in \underline{F}(\alpha) \\
& \text{ iff } (\{ \underline{F}(\beta) \in \underline{F}(\alpha) \}, \emptyset) \in G.
\end{aligned}$$

Thus $\mathring{F}(\alpha)^G \subseteq F_G(\alpha)$. For the converse inclusion, note as well that if $a \in F_G(\alpha)$, then $a = F_G(\beta)$, for some $\beta < \alpha$, because $\langle \mathbb{V}_G; F_G, \alpha \rangle_{\alpha < \kappa} \models (\dagger)$. $\square$

LEMMA 2.5. *If $\mathring{a}$ is a Shoenfield term in $L_\kappa^\mathbb{P}$ and G is $\mathbb{P}$ generic for $\langle L_\kappa; \mathbb{P} \rangle$ -definable predense classes, then $\mathring{a}^G \in \mathbb{V}_G$. Thus $L_\kappa[G] \subseteq \mathbb{V}_G$.*

PROOF: Let us call a Shoenfield term $\mathring{a}$ *special* when, hereditarily, only conditions of the form $(r, \emptyset)$ occur in $\mathring{a}$. That is,

$$\mathring{a} \text{ is special} \text{ iff } \vec{s} = \emptyset \text{ and } \mathring{b} \text{ is special, for all } (\mathring{b}, (r, \vec{s})) \in \mathring{a}.$$

The following two claims suffice to prove the lemma.

CLAIM. *If $\mathring{a}$ is a special Shoenfield term, then $\mathring{a}^G \in \mathbb{V}_G$.*

PROOF: Fix a special Shoenfield term $\mathring{a}$. Because $\mathbb{P}$ is L_κ -amenable,

$$\left\{ \mathring{b} \in \text{TC}(\{\mathring{a}\}) : \mathring{b} \text{ is a special Shoenfield term} \right\} \in L_\kappa,$$

where $\text{TC}(\{\dot{a}\})$ is the transitive closure of $\{\dot{a}\}$. Now

$$\dot{b}^G = \left\{ \dot{c}^G : \dot{c} \in \text{dom}(\dot{b}) \text{ and } \dot{b}(\dot{c}) \cap G \neq \emptyset \right\},$$

and for $\dot{c} \in \text{dom}(\dot{b})$,

$$\dot{b}(\dot{c}) \cap G \neq \emptyset \quad \text{iff} \quad \langle \mathbb{V}_G, F_G, \alpha \rangle_{\alpha < \kappa} \models \bigwedge r, \text{ for some finite set } r \\ \text{of basic sentences such that } (r, \emptyset) \in \dot{b}(\dot{c}).$$

Using that $\langle \mathbb{V}_G, F_G, \alpha \rangle_{\alpha < \kappa}$ satisfies ZFC in the language $\{\in, \underline{F}\} \cup \{\underline{\alpha} : \alpha < \kappa\}$, it follows by induction on special terms in $\text{TC}(\{\dot{a}\})$ that $\dot{a}^G \in \mathbb{V}_G$. $\square$

CLAIM. Suppose that $\dot{a}$ is a Shoenfield term and that G is $\langle L_\kappa; \mathbb{P} \rangle$ -definably generic. Then there exists a special Shoenfield term $\dot{a}^*$ such that $\dot{a}^G = \dot{a}^{*G}$.

PROOF: Fix a Shoenfield term $\dot{a}$. It suffices to see that there exists an $\langle L_\kappa; \mathbb{P} \rangle$ -definable dense class of conditions p'' such that for some special Shoenfield term $\dot{a}^*$, if G is $\langle L_\kappa; \mathbb{P} \rangle$ -definably generic and $p'' \in G$, then $\dot{a}^G = \dot{a}^{*G}$.

Pick any condition $p = (r, \vec{s})$. We may assume that $|\vec{s}| > 0$. Say $|\vec{s}| = n$ and set $\bar{\gamma} = \gamma_{n-1}$. Choose $\gamma \in X$ such that $\gamma \geq \|r\|$, $\bar{\gamma}$ and also $\gamma > \text{TC}(\{\dot{a}\}) \cap \text{OR}$.

Set $p'' = (r, \vec{s}'')$, where $\vec{s}'' = \vec{s} \smallfrown (j_{\bar{\gamma}\gamma}(s_{n-1}), \gamma)$. Then $p'' \leq p$ in $\mathbb{P}$. Furthermore, by our choice of γ and definition of $\vec{s}''$, if $(r', \vec{s}') \in \text{TC}(\{\dot{a}\})$ is compatible with p'' , then the sequence $\vec{s}'$ is an initial segment of the sequence $\vec{s}''$.

For Shoenfield terms $\dot{b}$ in $\text{TC}(\{\dot{a}\})$, set

$$\dot{b}^* = \left\{ (\dot{c}^*, (r', \emptyset)) : (\dot{c}, (r', \vec{s}')) \in \dot{b} \text{ and } (r', \vec{s}') \text{ is compatible with } p'' \right\}.$$

Then $\dot{b}^*$ is a special term, and if $q \leq p''$, then

$$q \leq (r', \vec{s}') \in \dot{b}(\dot{c}), \text{ for some } \vec{s}' \quad \text{iff} \quad q \leq (r', \emptyset) \in \dot{b}^*(\dot{c}^*).$$

It follows by induction on $\dot{b}$ in $\text{TC}(\{\dot{a}\})$ that if G is $\langle L_\kappa; \mathbb{P} \rangle$ -definably generic and $p'' \in G$, then $\dot{b}^G = \dot{b}^{*G}$. $\square$

This completes the proof of Theorem 2. $\square$

3. Two remarks

This section contains two remarks on Theorem 2. The first of these is that under the hypotheses of Theorem 2, the theory $T + \theta_\kappa$ has 2^ω many distinct invisibly generic models iff $T + \theta_\kappa \not\models V=L$.

The reader may recall that T is part of θ_κ . Nevertheless, we shall write ' $T + \theta_\kappa$ ' where ' θ_κ ' would suffice to render vividly that we are working with a certain infinitary extension of T .

Our second remark is that the forcing relation for $\mathbb{P}_T$ is uniformly definable over $\langle L_\kappa; \mathbb{P}_T \rangle$.

Suppose that the countable ordinal κ is definably regular in $\mathbb{H}(L_\kappa)$, that $T \in L_\kappa$ is a theory in the language of set theory including ZFC, and that $T + \theta_\kappa$ is satisfiable. It is not difficult to see that the second order predicate “ G is $\mathbb{P}_T$ generic for $\langle L_\kappa; \mathbb{P}_T \rangle$ -definable predense classes” is Σ_1^1 over $\mathbb{H}(L_\kappa)$. It follows that either every such G lies in $\mathbb{H}(\mathbb{H}(L_\kappa))$ or else there exist perfectly many such G ’s, c.f. [B, Theorem IV.4.4]. Using an idea of H. Friedman’s [HF, Theorem 6.2], though, we can prove a sharper result.

REMARK 3.1. *Suppose that the countable ordinal κ is definably regular in $\mathbb{H}(L_\kappa)$, that $T \in L_\kappa$ is a theory in the language of set theory including ZFC, and that $T + \theta_\kappa$ is satisfiable.*

- (1) *If $T + \theta_\kappa \models V=L$, then any two models of $T + \theta_\kappa$ are isomorphic in the language of set theory.*
- (2) *If $T + \theta_\kappa \not\models V=L$, then there exist 2^ω many pairwise non-isomorphic (in the language of set theory) invisibly generic models of $T + \theta_\kappa$.*

PROOF: Claim (1) is clear. For (2), let $\mathbb{V}$ be a model of $T + \theta_\kappa + V \neq L$. Say $\mathbb{V}$ satisfies φ , where the sentence φ formalizes “ $\underline{F}(\underline{\alpha}) \subseteq \underline{\lambda}$ and $\underline{F}(\underline{\alpha}) \not\subseteq L$,” for some α and λ .

CLAIM. *Suppose that $(r, \vec{s}) \in \mathbb{P}_T$ is such that $n = |\vec{s}| > 0$ and $\varphi \in s_{n-1}$. Then there exists a $\delta < \lambda$ such that both $(r \cup \{\delta \in \underline{F}(\underline{\alpha})\}, \vec{s})$ and $(r \cup \{\delta \notin \underline{F}(\underline{\alpha})\}, \vec{s})$ are conditions extending $(r, \vec{s})$ in $\mathbb{P}_T$.*

PROOF: Otherwise, set

$$x = \left\{ \delta < \lambda : (r \cup \{\delta \in \underline{F}(\underline{\alpha})\}, \vec{s}) \in \mathbb{P}_T \right\}.$$

Set $\gamma = \gamma_{n-1}$. Then $\delta \in x$ iff the infinitary theory $T \cup \{\theta_\kappa\} \cup r \cup j_{\gamma_\kappa}(s_{n-1}) \cup \{\delta \in \underline{F}(\underline{\alpha})\}$ is consistent in $\mathcal{L}_{\mathbb{H}(L_\kappa)}$. Thus x is Π_1 -definable over $\mathbb{H}(L_\kappa)$ and $x \subseteq \lambda$, which is less than κ , the Σ_1 -projectum. Consequently $x \in \mathbb{H}(L_\kappa)$. Again, since $x \subseteq \lambda < \kappa$, it follows that $x \in L_\kappa$.

Now if G is $\mathbb{P}_T$ generic and $(r, \vec{s}) \in G$, then $\underline{F}(\underline{\alpha})^{\mathbb{V}_G} = x$ and so $\mathbb{V}_G \models “\underline{F}(\underline{\alpha}) \in L,”$ contradicting that $\mathbb{V}_G \models \varphi$. $\square$

It follows that for each of 2^ω many different $x \subseteq \lambda$, there exists an $\langle L_\kappa; \mathbb{P}_T \rangle$ -definably generic G_x such that $x \in L_\kappa[G_x]$. Furthermore, each cell is countable when the set of such $L_\kappa[G_x]$ ’s is partitioned by isomorphism in the language of set theory, because each $L_\kappa[G_x]$ is countable. Consequently, there exist 2^ω many pairwise non-isomorphic such $L_\kappa[G_x]$. $\square$

We turn now to our second remark, namely, that the forcing relation for $\mathbb{P}_T$ is definable. This is perhaps surprising since predensity reduction fails for $\mathbb{P}_T$ and, in fact, generic classes G blatantly ω -cofinalize the ordinals of L_κ . It also shows that what is at issue in deciding whether $0^\#$ can be weakly generic is not the definability of the forcing relation.

Originally, I proved that the forcing relation for $\mathbb{P}_T$ is definable over $\langle L_\kappa; \mathbb{P}_T \rangle$, when it restricted to formulas of bounded logical complexity. David Cook extended that argument to show that, in fact, the forcing relation is uniformly definable for all sentences of the forcing language, noting that this does not immediately imply

that $L[G]$ -truth is uniformly definable over $L[G]$, because $\mathbb{P}_T$ and G are not definable over $L[G]$.

Even at bit more is true. In the sense made precise by Remark 3.2, the forcing relation is uniformly definable for all sentences of an infinitary forcing language with formulas lying in $\mathbb{H}(L_\kappa)$.

Fix T and set $\mathbb{P} = \mathbb{P}_T$. The ordinary *forcing language* for $\mathbb{P}$ is the finitary first order language without equality having the following non-logical symbols: two-place relation symbols $\in$ and $\subseteq$; and constant symbols $\dot{a}$, for each Shoenfield term $\dot{a} \in L_\kappa^\mathbb{P}$. As usual, we systematically confuse symbols and formulas with sets coding them in L_κ . In the forcing language we treat equality as a defined relation; that is, $t \approx t'$ abbreviates $(t \subseteq t') \wedge (t' \subseteq t)$.

In contrast to visible genericity, the standard structure $L_\kappa[G]$ cannot uniformly define satisfaction for atomic sentences of the forcing language, because the classes $L_\kappa^\mathbb{P}$ and G are not available. Working outside of $L_\kappa[G]$, declare that $L_\kappa[G] \models \dot{b} \in \dot{a}$ and $L_\kappa[G] \models \dot{a} \subseteq \dot{b}$ when $\dot{b}^G \in \dot{a}^G$ and when $\dot{a}^G \subseteq \dot{b}^G$, respectively. Then extend $\models$ to non-atomic sentences as usual.

Recall from Lemma 2.5 that special Shoenfield terms are terms that, hereditarily, contain only conditions of the form $(r, \emptyset)$. Fix any $\gamma \in X$ and work in $\mathbb{H}(L_\gamma)$. Say that $\dot{a}$ is a *special pseudo-term* when $\dot{a} \in L_\gamma$ and $\dot{a}$ is a set of pairs $(\dot{b}, (r, \emptyset))$, where r is any finite set of basic $\mathcal{L}_{\mathbb{H}(L_\gamma)}$ -sentences and $\dot{b}$ is a special pseudo-term. In particular, consistency of r with θ_γ is not required, so special pseudo-terms are not Shoenfield terms, in general. Of course, every special term is a special pseudo-term. The advantage that this notion enjoys is that $\{\dot{a} \in L_\gamma : \dot{a} \text{ is a special pseudo-term}\}$ is always a set in $\mathbb{H}(L_\gamma)$.

Next, let us define an infinitary forcing language $\mathcal{K}_{\mathbb{H}(L_\gamma)}$. Let $\mathcal{K}$ comprise the symbols $\in$, $\subseteq$, and $\dot{a}$, for all special pseudo-terms $\dot{a}$. Let $\mathcal{K}_{\mathbb{H}(L_\gamma)}$ be the usual infinitary language, namely, the intersection of $\mathcal{K}_{\infty\omega}$ with $\mathbb{H}(L_\gamma)$. Then $\mathcal{K}_{\mathbb{H}(L_\gamma)}$ is Δ_1 -definable over $\mathbb{H}(L_\gamma)$. Consequently, $j_{\gamma\kappa}(\varphi)$ is a formula in $\mathcal{K}_{\mathbb{H}(L_\kappa)}$ whenever φ is a formula in $\mathcal{K}_{\mathbb{H}(L_\gamma)}$. Furthermore, $\mathcal{K}_{\mathbb{H}(L_\kappa)} = \bigcup_{\gamma \in X \cap \kappa} j_{\gamma\kappa}'' \mathcal{K}_{\mathbb{H}(L_\gamma)}$.

Say that a formula φ of $\mathcal{K}_{\mathbb{H}(L_\gamma)}$ is *legitimate* when every special pseudo-term occurring in φ is a genuine (special) Shoenfield term in $L_\gamma^{\mathbb{P} \cap L_\gamma}$.

REMARK 3.2. *There exists an $\langle L_\kappa; \mathbb{P} \rangle$ -definable three-place relation $\Vdash$ (defined on triples (p, γ, φ) such that $p \in \mathbb{P}$, $\gamma \in X \cap \kappa$, and φ is a legitimate $\mathcal{K}_{\mathbb{H}(L_\gamma)}$ -sentence) having the property that if G is an $\langle L_\kappa; \mathbb{P} \rangle$ -definably generic filter on $\mathbb{P}$, then*

$$L_\kappa[G] \models j_{\gamma\kappa}(\varphi) \quad \text{iff} \quad \exists p \in G \, p \Vdash_\gamma \varphi,$$

whenever $\gamma \in X \cap \kappa$ and φ is a legitimate $\mathcal{K}_{\mathbb{H}(L_\gamma)}$ -sentence.

Of course, if φ is a finitary legitimate $\mathcal{K}_{\mathbb{H}(L_\kappa)}$ -sentence, then $\varphi \in L_\gamma$, for any sufficiently large $\gamma \in X \cap \kappa$, and $j_{\gamma\kappa}(\varphi) = \varphi$. Because in a generic extension each Shoenfield term is forced equal to some special Shoenfield term (in the sense of the second Claim in Lemma 2.5), it follows that the forcing relation for first order sentences of the ordinary forcing language is uniformly definable.

PROOF: Fix any $\gamma \in X$. Working in $\mathbb{H}(L_\gamma)$, define by recursion an operation associating a Boolean combination of basic $\mathcal{L}_{\mathbb{H}(L_\gamma)}$ -sentences to each $\mathcal{K}_{\mathbb{H}(L_\gamma)}$ -sentence:

$$\begin{aligned} \llbracket \dot{b} \in \dot{a} \rrbracket^\gamma &= \bigvee_{\dot{c} \in \text{dom}(\dot{a})} \left(\left(\bigvee_{(r, \emptyset) \in \dot{a}(\dot{c})} \bigwedge r \right) \wedge \llbracket \dot{b} \approx \dot{c} \rrbracket^\gamma \right) \\ \llbracket \dot{a} \subseteq \dot{b} \rrbracket^\gamma &= \bigwedge_{\dot{c} \in \text{dom}(\dot{a})} \left(\left(\bigvee_{(r, \emptyset) \in \dot{a}(\dot{c})} \bigwedge r \right) \rightarrow \llbracket \dot{c} \in \dot{b} \rrbracket^\gamma \right) \\ \llbracket \bigwedge \Phi \rrbracket^\gamma &= \bigwedge_{\varphi \in \Phi} \llbracket \varphi \rrbracket^\gamma \\ \llbracket \bigvee \Phi \rrbracket^\gamma &= \bigvee_{\varphi \in \Phi} \llbracket \varphi \rrbracket^\gamma \\ \llbracket \neg \varphi \rrbracket^\gamma &= \neg \llbracket \varphi \rrbracket^\gamma \\ \llbracket \forall v \varphi \rrbracket^\gamma &= \bigwedge_{\alpha < \gamma} \llbracket \varphi(\dot{F}(\alpha)/v) \rrbracket^\gamma, \end{aligned}$$

where $\dot{F}$ is as in Lemma 2.4.

CLAIM. Suppose that G is an $\langle L_\kappa; \mathbb{P} \rangle$ -definably generic filter on $\mathbb{P}$. Then

$$L_\kappa[G] \models \varphi \quad \text{iff} \quad \llbracket \varphi \rrbracket^\kappa \in \Gamma,$$

for all legitimate sentences φ of $\mathcal{K}_{\mathbb{H}(L_\kappa)}$.

PROOF: Proceed by induction on the complexity of φ to unwind definitions, using properties (Γ1)–(Γ8) and that every element of $L_\kappa[G]$ is of the form $\dot{F}(\alpha)^G$, for some $\alpha < \kappa$. Let us illustrate this with two cases:

$$\begin{aligned} L_\kappa[G] \models \dot{b} \in \dot{a} &\quad \text{iff} \quad \dot{b}^G \in \dot{a}^G \\ &\quad \text{iff} \quad \exists \dot{c} \in \text{dom}(\dot{a}) \left(\dot{a}(\dot{c}) \cap G \neq \emptyset \text{ and } \dot{b}^G = \dot{c}^G \right) \\ &\quad \text{iff} \quad \exists \dot{c} \in \text{dom}(\dot{a}) \left(\exists (r, \emptyset) \in \dot{a}(\dot{c}) \ (r, \emptyset) \in G \text{ and } \dot{b}^G = \dot{c}^G \right) \\ &\quad \text{iff} \quad \exists \dot{c} \in \text{dom}(\dot{a}) \left(\exists (r, \emptyset) \in \dot{a}(\dot{c}) \ \bigwedge r \in \Gamma \text{ and } \llbracket \dot{b} \approx \dot{c} \rrbracket^\kappa \in \Gamma \right) \\ &\quad \text{iff} \quad \bigvee_{\dot{c} \in \text{dom}(\dot{a})} \left(\left(\bigvee_{(r, \emptyset) \in \dot{a}(\dot{c})} \bigwedge r \right) \wedge \llbracket \dot{b} \approx \dot{c} \rrbracket^\kappa \right) \in \Gamma \\ &\quad \text{iff} \quad \llbracket \dot{b} \in \dot{a} \rrbracket^\kappa \in \Gamma. \\ L_\kappa[G] \models \forall v \varphi &\quad \text{iff} \quad L_\kappa[G] \models \varphi(\dot{F}(\alpha)/v), \text{ for all } \alpha < \kappa \\ &\quad \text{iff} \quad \llbracket \varphi(\dot{F}(\alpha)/v) \rrbracket^\kappa \in \Gamma, \text{ for all } \alpha < \kappa \\ &\quad \text{iff} \quad \bigwedge_{\alpha < \kappa} \llbracket \varphi(\dot{F}(\alpha)/v) \rrbracket^\kappa \in \Gamma \\ &\quad \text{iff} \quad \llbracket \forall v \varphi \rrbracket^\kappa \in \Gamma. \quad \square \end{aligned}$$

Using that $\llbracket \varphi \rrbracket^\gamma$ is defined over $\mathbb{H}(L_\gamma)$ by Σ_1 -recursion, note that if $\gamma \in X \cap \kappa$, then

$$j_{\gamma\kappa}(\llbracket \varphi \rrbracket^\gamma) = \llbracket j_{\gamma\kappa}(\varphi) \rrbracket^\kappa.$$

It follows that if φ is a legitimate sentence in $\mathcal{K}_{\mathbb{H}(L_\kappa)}$, then

$$\begin{aligned} \llbracket \varphi \rrbracket^\kappa \in \Gamma &\quad \text{iff} \quad \llbracket j_{\gamma_{n-1}\kappa}^{-1}(\varphi) \rrbracket^{\gamma_{n-1}} \in s_{n-1}, \\ &\quad \text{for some } (r, \vec{s}) \in G \text{ with } 0 < |\vec{s}| = n. \end{aligned}$$

Working in $\langle L_\kappa; \mathbb{P} \rangle$ define the three-place relation $\Vdash$ by declaring that $p \Vdash_\gamma \varphi$ iff $p \in \mathbb{P}$ and $\gamma \in X \cap \kappa$ and φ is a legitimate sentence of $\mathcal{K}_{\mathbb{H}(L_\gamma)}$ and

$$\left\{ (r, \vec{s}) : 0 < |\vec{s}| = n \text{ and } \gamma_{n-1} \geq \gamma \text{ and } j_{\gamma\gamma_{n-1}}(\llbracket \varphi \rrbracket^\gamma) \in s_{n-1} \right\}$$

is predense with respect to p .

Then $\Vdash$ is as required. $\square$

4. The regularity of $o(\mathbb{V})$

Theorem 2's most curious hypothesis is that κ is definably regular in $\mathbb{H}(L_\kappa)$; the most disappointing feature of its proof is that G collapses κ . It is natural to ask whether these are essential features of Theorem 2, or merely accidental ones of the proof given in §2.

The first of these questions can be given a fairly satisfactory answer without much trouble. In the proof only Σ_n -definable regularity, for sufficiently large n , is used. On the other hand, κ must be regular in $\mathbb{H}(L_\kappa)$ in the usual sense that no function that is an element of $\mathbb{H}(L_\kappa)$ renders κ singular:

PROPOSITION 4.1. *Suppose that κ is a countable ordinal and that there exists a real x such that $L_\kappa[x]$ satisfies $\text{ZFC} + \text{"}x \text{ is } 0^\#\text{"}$. Then κ is regular in $\mathbb{H}(L_\kappa)$.*

PROOF: Fix such a real x and let " $0^\#$ " denote it. Let δ be the least admissible ordinal greater than κ , so $L_\delta = \mathbb{H}(L_\kappa)$. Let $\mathfrak{A}$ be the Ehrenfeucht-Mostowski $(0^\#, \kappa + \omega)$ -model. Identify $\text{wf}(\mathfrak{A})$, the well-founded part of $\mathfrak{A}$, with the transitive set to which it is isomorphic. Now $\text{wf}(\mathfrak{A})$ must be an admissible set [B, Truncation Lemma] and $o(\text{wf}(\mathfrak{A})) > \kappa$ because a well ordering of type κ lies in $\mathfrak{A}$. Thus $L_\delta \subseteq \mathfrak{A}$, though possibly $\delta = o(\text{wf}(\mathfrak{A}))$.

Suppose now that $f: \lambda \rightarrow \kappa$, for some $\lambda < \kappa$, and that $f \in L_\delta$, hence $f \in \mathfrak{A}$. We maintain that the range of f is bounded in κ . Say $f = t^{\mathfrak{A}}(\iota_1, \dots, \iota_n, \gamma_1, \dots, \gamma_k)$, where t is a term and $\iota_1 < \dots < \iota_n < \kappa$ and $\kappa = \gamma_1 <^{\mathfrak{A}} \dots <^{\mathfrak{A}} \gamma_k$ are indiscernibles in the sense of $\mathfrak{A}$. (Of course, some of the γ_i 's need not be genuine ordinals.) Let $\bar{\gamma}_1 < \dots < \bar{\gamma}_k$ be indiscernibles in the sense of $\mathfrak{A}$ such that $\lambda, \iota_n < \bar{\gamma}_1 < \kappa$. Set $\bar{f} = t^{\mathfrak{A}}(\iota_1, \dots, \iota_n, \bar{\gamma}_1, \dots, \bar{\gamma}_k)$. Then $\mathfrak{A} \models \bar{f}: \lambda \rightarrow \bar{\gamma}_1$. But $\mathfrak{A} \models \bar{f}(\alpha) = f(\alpha)$, for each $\alpha < \lambda$ by indiscernibility. Hence $\text{rng}(f) \subseteq \bar{\gamma}_1 < \kappa$. $\square$

We cannot give as satisfactory an answer regarding the extent to which κ must lose its regularity. Note that $G \in \mathbb{H}(\langle L_\kappa; \mathbb{P}_T \rangle)[G]$ in the proof of Theorem 2, since $\mathring{G} = \{ (\check{p}, p) : p \in \mathbb{P}_T \} \in \mathbb{H}(\langle L_\kappa; \mathbb{P}_T \rangle)$. [Incidentally, $\mathbb{H}(\langle L_\kappa; \mathbb{P}_T \rangle) = \mathbb{H}(\mathbb{H}(L_\kappa)) = L_\delta$, where δ is the second admissible ordinal above κ .] It follows that κ has definable cofinality ω in $\mathbb{H}(\langle L_\kappa; \mathbb{P}_T \rangle)[G]$, indeed, has cofinality ω , if $\mathbb{H}(\langle L_\kappa; \mathbb{P}_T \rangle)[G]$ satisfies Σ_0 -separation.

Only with an additional hypothesis—and some effort—can we give an upper bound on how long $o(\mathbb{V})$ can remain regular.

Say that a filter $G \subseteq \mathbb{P}$ is *hyper-generic* over the countable standard transitive model $\langle \mathbb{V}; \mathbb{P} \rangle$ when $G \cap D \neq \emptyset$, for all predense D that are definable over $\mathbb{H}(\langle \mathbb{V}; \mathbb{P} \rangle)$. Of course if G is hyper-generic over $\langle \mathbb{V}; \mathbb{P} \rangle$, then G is $\langle \mathbb{V}; \mathbb{P} \rangle$ -definably generic.

If $\mathbb{V}[G] \models \text{ZFC} + \text{"}0^\# \text{ exists,}"$ let $0_G^\#$ be $(0^\#)^{\mathbb{V}[G]}$.

PROPOSITION 4.2. *Suppose that $\langle \mathbb{V}; \mathbb{P} \rangle$ is a countable standard transitive model of $ZFC + “0^\# \text{ does not exist},”$ that G is hyper-generic over $\langle \mathbb{V}; \mathbb{P} \rangle$, and that $\mathbb{V}[G]$ satisfies $ZFC + “0^\# \text{ exists}.”$ Then $o(\mathbb{V})$ is not definably regular in $\mathbb{H}(\langle \mathbb{V}; \mathbb{P} \rangle)[G]$.*

The proof will be recycled in the next section to prove Proposition 5.1.

PROOF: Set $\lambda = o(\mathbb{V})$. Note that $\lambda, \mathbb{V}, \mathbb{P} \in \mathbb{H}(\langle \mathbb{V}; \mathbb{P} \rangle)$. Because $\mathbb{P} \in \mathbb{H}(\langle \mathbb{V}; \mathbb{P} \rangle)$ and G is hyper-generic, we have that $\mathbb{H}(\langle \mathbb{V}; \mathbb{P} \rangle)[G]$ is admissible. (This is the only use made of G 's hyper-genericity.) We also have that $G \in \mathbb{H}(\langle \mathbb{V}; \mathbb{P} \rangle)[G]$.

The proof comes in three parts. Assuming for a contradiction that λ is definably regular in $\mathbb{H}(\langle \mathbb{V}; \mathbb{P} \rangle)[G]$, Part I reflects a long definable well-founded relation down into the universe $\mathbb{V}[G]$. Part II uses this to show that then $0^\#$ must lie in $\mathbb{V}$. The argument in Part II uses that if κ is a cardinal of uncountable cofinality in $L[0^\#]$, then any club subset of κ that is eventually contained in every constructible club subset of κ constructs $0^\#$. This is proved in Part III.

Part I. Suppose for a contradiction that λ is definably regular in $\mathbb{H}(\langle \mathbb{V}; \mathbb{P} \rangle)[G]$. Then there exists a Σ_2 -elementary substructure X of $\mathbb{H}(\langle \mathbb{V}; \mathbb{P} \rangle)[G]$ such that $\lambda, \mathbb{V}, \mathbb{P}, G \in X$ and $X \cap \mathbb{V}[G] = \mathbb{V}_\kappa[G]$, where $\kappa = X \cap \lambda$ is some limit cardinal of uncountable cofinality in $\mathbb{V}[G]$. Note that κ is inaccessible in $\mathbb{V}$, since $\mathbb{V} \models “0^\# \text{ does not exist}.”$

Set $\mathbb{A} = \langle \mathbb{V}_\kappa; \mathbb{P} \cap \mathbb{V}_\kappa \rangle$ and $G' = G \cap \mathbb{V}_\kappa$. Then $X \cong \mathbb{H}(\mathbb{A})[G']$ and $\mathbb{H}(\mathbb{A})[G']$ is admissible. [Checking this requires some calculations. Note that $\mathbb{H}(\mathbb{A})$ is Σ_1 definable over $\mathbb{H}(\mathbb{A})[G']$ and that $\mathbb{H}(\mathbb{A})[G] \models \forall x \exists \dot{a} (x = \dot{a}^G)$.]

Working in $\mathbb{V}_\kappa$, define

$$\text{OR}^{\mathbb{V}_\kappa} \cdot \omega = \left\{ (i, \alpha) : i \in \omega \text{ and } \alpha \text{ is an ordinal} \right\}$$

and let $\text{OR}^{\mathbb{V}_\kappa} \cdot \omega$ be ordered lexicographically (in type $\kappa \cdot \omega$). Set

$$B = \left\{ t(b_1, \dots, b_n) : t \text{ is a definable term of } ZFC + V=L \text{ and } b_1, \dots, b_n \in \text{OR}^{\mathbb{V}_\kappa} \cdot \omega \right\}.$$

In $\mathbb{V}_\kappa[G']$ define a relation E on B by

$$t(b_1, \dots, b_n) E t'(b'_1, \dots, b'_m) \quad \text{iff} \quad \ulcorner t(v_{i_1}, \dots, v_{i_n}) \in t'(v_{j_1}, \dots, v_{j_m}) \urcorner \in 0^\#_{G'},$$

where the sequence of natural numbers $\langle i_1, \dots, i_n, j_1, \dots, j_m \rangle$ is ordered just as the sequence $\langle b_1, \dots, b_n, b'_1, \dots, b'_m \rangle$ is ordered in $\text{OR}^{\mathbb{V}_\kappa} \cdot \omega$. Then $\langle B, E \rangle$ is a $\mathbb{V}_\kappa[G']$ -definable version of the Ehrenfeucht-Mostowski $(0^\#_G, \kappa \cdot \omega)$ -model.

Working in $\mathbb{V}[G]$ we have that $\langle B, E \rangle \cong L_\delta$, where δ is the $(\kappa \cdot \omega)^{\text{th}}$ Silver indiscernible provided by $0^\#_G$. Hence $\langle B, E \rangle$ is well-founded. Because $\langle B, E \rangle$ is definable over $\mathbb{V}_\kappa[G']$, we have that $\langle B, E \rangle \in \mathbb{H}(\mathbb{A})[G']$ and so $\delta < o(\mathbb{H}(\mathbb{A})[G'])$, since $\langle B, E \rangle$ is actually well-founded and δ is its rank. Furthermore, the isomorphism of $\langle B, E \rangle$ and L_δ lies in $\mathbb{H}(\mathbb{A})[G']$, by Δ_1 -separation.

Thus we have

- $o(\mathbb{H}(\mathbb{A})[G']) = o(\mathbb{H}(\mathbb{A}))$;
- $\kappa = o(\mathbb{A})$ is inaccessible in $\mathbb{H}(\mathbb{A})$ and is a limit cardinal of uncountable cofinality in $\mathbb{H}(\mathbb{A})[G']$; and
- $\langle B, E \rangle \cong L_\delta$, for some $\delta < o(\mathbb{H}(\mathbb{A}))$, and this isomorphism lies in $\mathbb{H}(\mathbb{A})[G']$.

In fact, $\mathbb{H}(\mathbb{A})[G'] \models “\kappa \text{ is inaccessible},”$ but we don't need this.

Part II. Choose $\triangleleft \in \mathbb{H}(\mathbb{A})$ to be a well-founded relation on $\mathbb{V}_\kappa$ having rank δ . Let $\rho_{\triangleleft} : \mathbb{V}_\kappa \rightarrow \delta$ be the rank function for $\triangleleft$:

$$\rho_{\triangleleft}(a) = \sup \left\{ \rho_{\triangleleft}(b) + 1 : b \triangleleft a \right\}$$

Then $\rho_{\triangleleft} \in \mathbb{H}(\mathbb{A})$ by Δ_1 -separation.

Define $\langle C_a : a \in \mathbb{V}_\kappa \rangle$ by letting C_a be the $\rho_{\triangleleft}(a)^{\text{th}}$ element of L_δ under $<_L$, if that element is a club subset of κ ; otherwise, let $C_a = \kappa$. Then $\langle C_a : a \in \mathbb{V}_\kappa \rangle \in \mathbb{H}(\mathbb{A})$ by Δ_1 -separation.

Define $\langle K_\alpha : \alpha < \kappa \rangle$ by setting $K_\alpha = \bigcap_{a \in \mathbb{V}_\alpha} C_a$. Again $\langle K_\alpha : \alpha < \kappa \rangle \in \mathbb{H}(\mathbb{A})$. Furthermore, K_α is a club subset of κ , for each $\alpha < \kappa$, because κ is inaccessible in $\mathbb{H}(\mathbb{A})$ and so $|\mathbb{V}_\alpha| < \kappa$. (Recall that $\mathbb{A} = \langle \mathbb{V}_\kappa; \mathbb{P} \cap \mathbb{V}_\kappa \rangle$, so the axiom of choice holds in $\mathbb{A}$.)

Set $K = \bigtriangleup_{\alpha < \kappa} K_\alpha$. Then the club set $K \subseteq \kappa$ lies in $\mathbb{H}(\mathbb{A})$ and is eventually contained in each club $C \subseteq \kappa$ such that $C \in L_\delta$.

Work in $\mathbb{H}(\mathbb{A})[G']$. Because the isomorphism of $\langle B, E \rangle$ with L_δ lies in $\mathbb{H}(\mathbb{A})[G']$, there exist Silver indiscernibles I for L_δ having order type $\kappa \cdot \omega$. Let $\langle \iota_\xi : \xi < \kappa \cdot \omega \rangle$ enumerate I monotonically and let the elementary embedding $j : L_\delta \rightarrow L_\delta$ be induced by $j(\iota_\xi) = \iota_{\kappa+\xi}$, for $\xi < \kappa \cdot \omega$.

If $\bar{C}$ is a club subset of ι_0 and lies in L_δ , then $j(\bar{C})$ is club in $\iota_\kappa = \kappa$ and lies in L_δ . Consequently, K is eventually contained in $j(\bar{C})$. Now $\mathcal{P}^{L_\delta}(\iota_0)$ is countable in $\mathbb{H}(\mathbb{A})[G']$ and κ has uncountable cofinality in $\mathbb{H}(\mathbb{A})[G']$. Thus there exists a $\beta < \kappa$ such that

$$K \setminus \beta \subseteq \bigcap \left\{ j(\bar{C}) : \bar{C} \in L_\delta \text{ is club in } \iota_0 \right\}.$$

It follows by Lemma 4.3.5 below that $K \setminus \beta \subseteq I \cap \kappa$. Hence $K \setminus \beta$ is an unbounded set of indiscernibles for L_κ . Since κ is inaccessible in $\mathbb{V}$ and since $K \in \mathbb{H}(\mathbb{A}) \subseteq \mathbb{V}$, it follows that $\mathbb{V} \models "0^\# \text{ exists}."$ $\square$

Part III: Fast closed unbounded sets

This final part of the proof of Proposition 4.2 is independent of the rest of this paper. It is devoted to proving the following elementary fact, which was used above in Part II:

PROPOSITION 4.3. *Suppose that $0^\#$ exists and that κ is a cardinal of uncountable cofinality in $L[0^\#]$. Suppose that C is a closed unbounded (club) subset of κ that is eventually contained in every constructible club subset of κ . Then $0^\# \in L[C]$.*

LEMMA 4.3.1. *Suppose that κ is inaccessible in L and that $\mathcal{U}$ is an L -ultrafilter on κ . Suppose that $f : \kappa \rightarrow \kappa$ is a constructible function such that $\{ \alpha < \kappa : f(\alpha) > \alpha \} \in \mathcal{U}$. Then there exists a constructible club $C \subseteq \kappa$ such that $\{ \alpha < \kappa : f(\alpha) \notin C \} \in \mathcal{U}$.*

PROOF: Set $C = \{ \beta < \kappa : f''\beta \subseteq \beta \}$. Then C is constructible club subset of κ and $\{ \alpha < \kappa : f(\alpha) \in C \} \subseteq \{ \alpha < \kappa : f(\alpha) \leq \alpha \} \notin \mathcal{U}$. $\square$

LEMMA 4.3.2. *Suppose that $j: L \rightarrow L$ is elementary with critical point $\bar{\iota}$ and that $j(\bar{\iota}) = \iota$. If $C \subseteq \iota$ is a constructible club set, then there exists a constructible club $\bar{C} \subseteq \bar{\iota}$ such that $j(\bar{C})$ is eventually contained in C .*

PROOF: Let I be the class of Silver indiscernibles provided by j and suppose that $C \in \text{Skolem Hull}_L(\iota \cup \{\iota, \delta_1, \dots, \delta_k\})$, where $\delta_1, \dots, \delta_k \in I \setminus (\iota + 1)$. Let $\langle \bar{D}_\alpha : \alpha < \bar{\iota} \rangle$ be a constructible enumeration of

$$\left\{ \bar{D} \in \text{Skolem Hull}_L(\bar{\iota} \cup \{\bar{\iota}, \delta_1, \dots, \delta_k\}) : \bar{D} \subseteq \bar{\iota} \text{ is club} \right\}.$$

Set $\bar{C} = \Delta_{\alpha < \bar{\iota}} \bar{D}_\alpha$. Then $j(\bar{C}) = \Delta_{\alpha < \iota} D_\alpha$, where $\langle D_\alpha : \alpha < \iota \rangle = j(\langle \bar{D}_\alpha : \alpha < \bar{\iota} \rangle)$ is a constructible enumeration of

$$\left\{ D \in \text{Skolem Hull}_L(\iota \cup \{\iota, \delta_1, \dots, \delta_k\}) : D \subseteq \iota \text{ is club} \right\}. \quad \square$$

COROLLARY 4.3.3. *Suppose that $j: L \rightarrow L$ is elementary and that $\iota = j(\bar{\iota})$. Suppose that $f: \iota \rightarrow \iota$ is constructible and such that $\{\alpha < \iota : f(\alpha) > \alpha\} \in \mathcal{U}$, where $\mathcal{U}$ is an L -ultrafilter on ι . Then there exists a constructible club $\bar{C} \subseteq \bar{\iota}$ such that $\{\alpha < \iota : f(\alpha) \notin j(\bar{C})\} \in \mathcal{U}$.*

PROOF: If j is trivial, then the corollary just restates Lemma 4.3.1. Otherwise, use Lemma 4.3.2, as well. $\square$

Let us next introduce some notation. First of all, if $\bar{\iota} < \iota$ are indiscernibles, let $j_{\bar{\iota}\iota}$ be the embedding induced by omitting $I \cap [\bar{\iota}, \iota)$. That is, let $\langle \iota_\xi : \xi \in \text{OR} \rangle$ enumerate the Silver indiscernibles I monotonically; if $\bar{\iota} < \iota$ in I , say $\bar{\iota} = \iota_\alpha$ and $\iota = \iota_\beta$, let $j_{\bar{\iota}\iota}: L \rightarrow L$ be the elementary embedding induced by

$$j_{\bar{\iota}\iota}(\iota_\xi) = \begin{cases} \iota_\xi, & \text{if } \xi < \alpha; \\ \iota_{\beta+\zeta}, & \text{if } \xi = \alpha + \zeta. \end{cases}$$

Next, let us introduce some notation regarding L -ultrapowers. If $j: L \rightarrow L$ is any elementary embedding and κ is the critical point of j , then define a normal L -ultrafilter on κ by

$$X \in \mathcal{U}_j \quad \text{iff} \quad X \text{ is a constructible subset of } \kappa \text{ and } \kappa \in j(X).$$

Let $\text{Ult}(L, \mathcal{U}_j)$ be the usual definable class ultrapower of L by $\mathcal{U}_j$. Then $e: x \mapsto [\text{id}_x]$ is an elementary embedding $L \rightarrow \text{Ult}(L, \mathcal{U}_j)$, where $[\text{id}_x]$ is (a Scott representative of) the equivalence class of the function id_x with domain κ , taking the value x identically.

Also as usual, let $\sigma: \text{Ult}(L, \mathcal{U}_j) \rightarrow_{\Sigma_\omega} L$ by

$$\sigma([f]) = j(f)(\kappa).$$

Finally, let $\mu: \sigma'' \text{Ult}(L, \mathcal{U}_j) \cong L$ be the transitive collapse of the range of σ . Define $\hat{j}$ to be the compositions of these functions: $\hat{j} = \mu \circ \sigma \circ e$. Then $\hat{j}: L \rightarrow_{\Sigma_\omega} L$.

LEMMA 4.3.4. Suppose that $0^\#$ exists and that ι is an indiscernible. Let ι^* be the least indiscernible greater than ι and set $j = j_{\iota^*}$. Then $\hat{j} = j$.

PROOF: First note that for *any* non-trivial $j: L \rightarrow_{\Sigma_\omega} L$ we have that $j = \sigma \circ e$. Indeed, if the critical point of j is κ , then

$$\begin{aligned} (\sigma \circ e)(x) &= \sigma([\text{id}_x]) \\ &= j(\text{id}_x)(\kappa) \\ &= j(x). \end{aligned}$$

Thus when $j = j_{\iota^*}$ it suffices to see that μ is the identity function, that is, that $\text{rng}(\sigma) = L$. Now $\text{rng}(j) \subseteq \text{rng}(\sigma)$ by the above calculation. Thus $I \setminus \{\iota\} \subseteq \text{rng}(\sigma)$. It suffices to see that $\iota \in \text{rng}(\sigma)$. For this, we need a constructible $f: \iota \rightarrow \iota$ such that $j(f)(\iota) = \iota$. The diagonal function $f(\alpha) = \alpha$ is such a function. $\square$

LEMMA 4.3.5. Let I be the class of Silver indiscernibles. Suppose that $j: L \rightarrow L$ is non-trivial and elementary with critical point $\bar{\iota}$. Set $\iota = j(\bar{\iota})$ and

$$E = \bigcap \left\{ j(\bar{C}) : \bar{C} \subseteq \bar{\iota} \text{ is a constructible club} \right\}.$$

Then $E = I \cap [\bar{\iota}, \iota)$.

PROOF: If $\bar{C} \subseteq \bar{\iota}$ is a constructible club, then $\bar{\iota} \in j(\bar{C})$. It follows by indiscernibility that $I \cap [\bar{\iota}, \iota) \subseteq E$.

Given $\delta < \bar{\iota}$, the interval $(\delta, \bar{\iota})$ is club in $\bar{\iota}$ and $j((\delta, \bar{\iota})) = (\delta, \iota)$. Consequently $\delta \notin E$.

Finally, we must see that if $\delta \in (\bar{\iota}, \iota) \setminus I$, then $\delta \notin E$. Suppose that $\bar{\iota} \leq \iota' < \delta < \iota'' \leq \iota$, where ι'' is the least indiscernible greater than ι' .

Now $j: L \rightarrow_{\Sigma_\omega} L$ is induced by the order-preserving shift of the indiscernibles $j \upharpoonright I$. It follows that there exists an elementary $j': L \rightarrow L$ such that $j' \upharpoonright \iota = \text{id} \upharpoonright \iota$ and $j = j' \circ j_{\bar{\iota}}$. In turn, $j_{\bar{\iota}} = j_{\iota''\iota} \circ j_{\iota'\iota''} \circ j_{\bar{\iota}\iota'}$. Set $\mathcal{U} = \mathcal{U}_{j_{\iota'\iota''}}$. Then $j_{\iota'\iota''} = \sigma \circ e$, where

$$L \xrightarrow{e} \text{Ult}(L, \mathcal{U}) \xrightarrow[\text{onto}]{\sigma} L.$$

Choose $f \in L$ such that $f: \iota' \rightarrow \iota'$ and $\delta = \sigma([f])$. Since $\iota' < \delta$,

$$\left\{ \alpha < \iota' : f(\alpha) > \alpha \right\} \in \mathcal{U}.$$

By Corollary 4.3.3, there exists a constructible club $\bar{C} \subseteq \bar{\iota}$ such that

$$\left\{ \alpha < \iota' : f(\alpha) \notin j_{\bar{\iota}\iota'}(\bar{C}) \right\} \in \mathcal{U}.$$

Then

$$\text{Ult}(L, \mathcal{U}) \models [f] \notin e(j_{\bar{\iota}\iota'}(\bar{C})),$$

hence

$$L \models \delta \notin \sigma(e(j_{\bar{\iota}\iota'}(\bar{C}))) = j_{\bar{\iota}\iota''}(\bar{C}).$$

Now $j' \circ j_{\iota''\iota} \upharpoonright \iota'' = \text{id} \upharpoonright \iota''$. Thus $\delta \notin j'(j_{\iota''\iota}(j_{\bar{\iota}\iota''}(\bar{C}))) = j(\bar{C})$ and so $\delta \notin E$. $\square$

PROOF OF PROPOSITION 4.3: Work in $L[0^\#]$. Note that κ is an indiscernible, since κ is a cardinal in $L[0^\#]$. Let $\bar{\iota}$ be the least indiscernible and set $j = j_{\bar{\iota}\kappa}$. By hypothesis the club $C \subseteq \kappa$ is eventually contained in $j(\bar{C})$, for each constructible club $\bar{C} \subseteq \bar{\iota}$. But there are only countably many such $\bar{C}$'s and κ has uncountable cofinality. Consequently, there exists $\beta < \kappa$ such that

$$C \setminus \beta \subseteq \bigcap \left\{ j(\bar{C}) : \bar{C} \subseteq \bar{\iota} \text{ is a constructible club} \right\} = I \cap [\beta, \kappa).$$

It follows that $0^\# \in L[C]$. $\square$

5. Open problems

5.1. Is the real $0^\#$ generic?

In order to make such a question precise—lacking some greater metaphysics—we should seek to isolate some internally accessible property that decides whether $0^\#$ is generic. By “the real $0^\#$,” then, we mean the $0^\#$ of the ambient universe, perhaps assuming ancillary axioms.

Making sense of this project may well require that “internal accessibility” be understood generously. Class genericity is at best a second-order property. Furthermore, by Theorem 2, there is no first-order statement dictating that $0^\#$ is not invisibly generic. On the other hand, it is conceivable that the real $0^\#$ is generic for first-order reasons:

QUESTION A. *Suppose that $0^\#$ exists. Do there exist definable classes M , $\mathbb{P}$, and G such that*

- $\langle M; \mathbb{P} \rangle$ is an inner model of ZFC and $0^\# \notin M$;
- $G \subseteq \mathbb{P}$ is $\langle M; \mathbb{P} \rangle$ -definably generic; and
- $0^\# \in M[G]$?

Say that a real x is a “ $0^\#$ supported by $\mathbb{V}$,” where $\mathbb{V}$ is a countable standard transitive model of ZFC, when there exists a countable standard transitive model $\mathbb{V}' \supseteq \mathbb{V}$ such that $o(\mathbb{V}') = o(\mathbb{V})$, $x \in \mathbb{V}' \setminus \mathbb{V}$, and $\mathbb{V}' \models \text{ZFC} + “x \text{ is } 0^\#.”$ Call a $0^\#$ supported by a countable $\mathbb{V}$ a “countable $0^\#$.”

QUESTION B. *Under suitable hypotheses, is there a countable weakly generic $0^\#$?*

That is, does there exist a countable standard transitive model $\mathbb{V}$ and a $\mathbb{V}$ -amenable partial ordering $\mathbb{P}$ such that

- $\langle \mathbb{V}; \mathbb{P} \rangle \models \text{ZFC} + “0^\# \text{ does not exist;}”$
- $G \subseteq \mathbb{P}$ is $\langle \mathbb{V}; \mathbb{P} \rangle$ -definably generic; and
- $\langle \mathbb{V}[G]; \mathbb{V}, \mathbb{P}, G \rangle \models \text{ZFC} + “0^\# \text{ exists;}”$

Affirming A affirms B. In fact, if A is affirmed, then *every* countable $0^\#$ is weakly generic.

“Every countable $0^\#$ is weakly generic” might be seen to provide second-order grounds for claiming that the real $0^\#$ is generic.

QUESTION C. *Suppose that $\mathbb{V}$ is a countable standard transitive model of ZFC + “ $0^\#$ does not exist.” Is every $0^\#$ supported by $\mathbb{V}$ invisibly generic?*

Again, affirming A affirms C. In fact, “every countable $0^\#$ is weakly generic” affirms C.

Invisible genericity is not a second-order property (in set theory) because, as in the case of Theorem 2, ZFC may fail in $\langle \mathbb{V}[G]; \mathbb{V}, \mathbb{P}, G \rangle$. Nevertheless, affirming Question C would provide some grounds for claiming that the real $0^\#$ is generic.

“Good” countable $0^\#$ ’s are counterexamples to a strengthened C in which genericity is replaced by hyper-genericity:

PROPOSITION 5.1. *Suppose that*

- (1) $\langle \mathbb{V}; \mathbb{P} \rangle$ is a countable standard transitive model of ZFC + “ $0^\#$ does not exist;”
- (2) G is hyper-generic over $\langle \mathbb{V}; \mathbb{P} \rangle$;
- (3) $\mathbb{V}[G]$ satisfies ZFC + “ $0^\#$ exists;”
- (4) the forcing relation for $\mathbb{P}$, when restricted to bounded formulas, is definable over $\langle \mathbb{V}; \mathbb{P} \rangle$; and that
- (5) $o(\mathbb{V})$ has uncountable cofinality in $\mathbb{H}(\langle \mathbb{V}; \mathbb{P} \rangle)[G]$.

Then the Ehrenfeucht-Mostowski $(0_G^\#, o(\mathbb{V}) \cdot \omega)$ -model is not well-founded.

In fact, if $\kappa = o(\mathbb{V})$, then $(\kappa^+)^L$ as calculated by $0_G^\#$ is not well-founded.

Hypothesis (4) can be replaced by “ $\langle \mathbb{V}[G]; \mathbb{V}, \mathbb{P}, G \rangle$ satisfies ZFC,” because this implies the definability of the forcing relation on a cone of conditions lying in G .

PROOF: Set $\mathbb{A} = \langle \mathbb{V}; \mathbb{P} \rangle$ and $\kappa = o(\mathbb{V})$. We may assume that $\mathbb{P} \Vdash$ “ $0^\#$ exists” and that $o(\mathbb{V})$ has uncountable cofinality in $\mathbb{H}(\mathbb{A})[G]$ whenever G is hyper-generic over $\langle \mathbb{V}; \mathbb{P} \rangle$. We may also assume that $\mathbb{P}$ has a weakest condition, so that ordinary canonical terms $\check{a}$ for ground model sets a are defined.

Note first that no countable $0^\#$ supported by $\mathbb{V}$ lies in $\mathbb{H}(\mathbb{A})$. Indeed, if $a \in \mathbb{H}(\mathbb{A})$ is a countable $0^\#$ supported by $\mathbb{V}$, then we can reach a contradiction as follows. Working in $\mathbb{V}$, let $\check{x}$ be a term such that $\mathbb{P} \Vdash$ “ $\check{x}$ is $0^\#$ ”. We maintain that $a = \{ n : \mathbb{P} \Vdash \check{n} \in \check{x} \}$, in contradiction to hypothesis (1). Otherwise, let $p \Vdash \check{n} \in \check{x}$, for some $n \notin a$, or $p \Vdash \check{n} \notin \check{x}$, for some $n \in a$. Let G be hyper-generic with $p \in G$. Since G is *a fortiori* definably generic over $\langle \mathbb{V}; \mathbb{P} \rangle$, we have that $\check{x}^G \neq a$ and that $\check{x}^G$ is a countable $0^\#$ supported by $\mathbb{V}$. Furthermore, both $\check{x}^G$ and a lie in $\mathbb{H}(\mathbb{A})[G]$, contradicting that $o(\mathbb{V})$ has uncountable cofinality there.

Fix a hyper-generic G and, working in $\mathbb{A}[G]$, define the $(0_G^\#, \kappa \cdot \omega)$ -model $\langle B, E \rangle$ as in the proof of Proposition 4.2. If $\langle B, E \rangle$ is well-founded, then we have that

- $\mathbb{H}(\mathbb{A})[G]$ is admissible and $o(\mathbb{H}(\mathbb{A})[G]) = o(\mathbb{H}(\mathbb{A}))$;
- $\kappa = o(\mathbb{A})$ has uncountable cofinality in $\mathbb{H}(\mathbb{A})[G]$; and
- $\langle B, E \rangle \cong L_\delta$, for some $\delta < o(\mathbb{H}(\mathbb{A}))$, and the isomorphism lies in $\mathbb{H}(\mathbb{A})[G]$.

By Part II of the proof of Proposition 4.2 (with $G' = G$), it follows that $0_G^\# \in \mathbb{H}(\mathbb{A})$, contradicting that no countable $0^\#$ supported by $\mathbb{V}$ lies in $\mathbb{H}(\mathbb{A})$. $\square$

5.2. Non-generic reals

Whether or not $0^\#$ itself turns out to be weakly generic, we can ask

QUESTION D. *Suppose that $0^\#$ exists. Can there exist a non-constructible real x that is not weakly generic.*

Sy Friedman [SF] has found a real x such that $0 <_c x <_c 0^\#$ and x is not L -amenably generic.

Working over the minimum model of ZFC, [S1] produces a “diagonally generic” real that is not weakly generic. Of course, the minimum model does not support a $0^\#$. And the method of [S1] is tied to minimality.

Finally,

QUESTION E. *Do there exist non-constructible reals that are not invisibly generic?*

6. References

- [B] Jon Barwise, *Admissible Sets and Structures*, Perspectives in Mathematical Logic, Springer-Verlag, New York (1975).
- [HF] Harvey Friedman, *Countable models of set theory*, Cambridge Summer School in Math. Logic, Lecture Note in Math., vol. 337, Springer-Verlag, New York (1973), pp. 539–573.
- [SF] Sy Friedman, *The genericity conjecture*, Jour. Sym. Logic **59** (1994), pp. 606–614.
- [S1] M.C. Stanley, *A non-generic real incompatible with $0^\#$* , Annals of Pure and Applied Logic (to appear).

Added in proof. Subsequent work has shed some light on two of these questions. The answer to Question C is Yes, if the definition of “invisibly generic” is modified. Say that $V[G]$ is *generic* if, whenever $V[G] \models \varphi$ also $V[G'] \models \varphi$, for comeagerly many G' in a neighborhood of G in the Stone Space of the forcing partial ordering.

An observation on the result of [4] largely affirms Question E. The real of [4] is not invisibly generic over any inner model for any admissibility preserving forcing. “Admissibility preserving” means that $V[G]$, the class of values of Shoenfield terms, is a model of KP whenever G is definably generic.

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