OUTER MODELS AND GENERICITY

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1. Introduction

Why is forcing the only known method for constructing outer models of set theory?

If V is a standard transitive model of ZFC, then a standard transitive model W of ZFC is an **outer model** of V if $V \subseteq W$ and $V \cap \text{OR} = W \cap \text{OR}$.

Is every outer model of a given model a generic extension? At one point Solovay conjectured that if $0^\#$ exists, then every real that does not construct $0^\#$ lies in $L[G]$, for some G that is generic for some forcing $\mathbb{P} \in L$. Famously, this was refuted by Jensen's coding theorem. He produced a real that is generic for an L -definable class forcing property, but does not lie in any set forcing extension of L .

Beller, Jensen, and Welch in *Coding the universe* [BJW] revived Solovay's conjecture by asking the following question: *Let $a \subseteq \omega$ be such that $L[a] \models "0^\# \text{ does not exist}"$. Is there a $b \in L[a]$ such that $a \notin L[b]$ and a is set generic over $L[b]$.*

In [S1] it was shown that even if arbitrary inner models are allowed, rather than just ones of the form $L[b]$, and even if we allow a to be class generic, the answer is No in general:

Theorem 1.1. *Let L_α be a minimal countable standard transitive model of ZFC. There exists a real x_{nwg} having the following three properties:*

- (1) $x_{\text{nwg}} \notin L_\alpha$.
- (2) $L_\alpha[x_{\text{nwg}}] \models \text{ZFC}$.
- (3) x_{nwg} is not definably generic over any outer model of L_α that does not already contain x_{nwg} .

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A precise statement of (3) is the following: Assume that V is an outer model of L_α and that $\mathbb{P}$ is a V -amenable partial ordering such that $(V; \mathbb{P})$ satisfies ZFC. Assume that the forcing relation restricted to sentences of bounded complexity is definable over $(V; \mathbb{P})$. (See Remark 1.8 regarding this hypothesis.) If G is a maximal filter on $\mathbb{P}$ meeting every dense subclass of $\mathbb{P}$ that is definable over $(V; \mathbb{P})$ and $x_{\text{nwg}} \in V[G]$, then $x_{\text{nwg}} \in V$.

For the sake of clarity, an elementary remark is in order. As is customary, we write “ V ” for the standard structure $(V; \in)$. If $S \subseteq V$, then “ $(V; S)$ satisfies ZFC” means that $(V; S, \in)$ satisfies ZFC in an enlarged language with a predicate symbol for S . In this case the axioms of ZFC are augmented by instances of Collection and Separation formulated in the enlarged language. We call this extended theory “ZFC” as well, relying on context to indicate the language. Similarly, we sometimes consider ZFC in a language augmented with many predicate symbols or function symbols. Augmenting the language of set theory with constant symbols does not increase the strength of ZFC.

Theorem 1.1 shows that, technically, forcing is *not* the only known method for constructing outer models of set theory. But the setting of this theorem, namely, a minimal model of $V = L$, is highly specialized. Are there such thoroughly non-generic extensions more generally? Sy Friedman [F1] proved that there exists a real x_{nag} such that $0 <_c x_{\text{nag}} <_c 0^\#$ and x_{nag} is not definably generic for any L -amenable class forcing. But x_{nag} is generic over an outer model of L that does not contain x_{nag} .

This paper proves a result complementary to Theorem 1.1, namely, that every outer model of a sufficiently non-minimal universe is a generic extension of that universe with respect to the language of set theory. (Remark 1.3 explains this qualification.)

Let us give a more precise statement of this result. Assume that there exists a global well-ordering of the universe V , so that we can calculate Skolem hulls uniformly. If, for example, $V = L$, then such a well-ordering is definable. In any case, an amenable well-ordering of the universe can be added by a simple class forcing and we can work relative to a predicate for this well-ordering.

If X is a set, let $\mathbf{Hyp}(X)$ be the smallest admissible set with X as an element. Work in V . If κ is a cardinal, let H_κ be the set of all sets of hereditary cardinality less than κ . Set

$$u(\kappa) = \left\{ \lambda \leq \kappa : \lambda \text{ is a cardinal and } H_\lambda = \text{Skolem Hull}_{\mathbf{Hyp}(H_\kappa)}(H_\lambda) \cap H_\kappa \right\}.$$

Then $u(\kappa) \subseteq \kappa + 1$ is a closed set of cardinals with $\kappa \in u(\kappa)$. Set

$$U = \left\{ u(\kappa) : \kappa \text{ is a cardinal} \right\}.$$

Then U is a definable tree under end-extension, but has no definable cofinal branches in V .

Theorem 1.2. *Suppose that W is a countable standard transitive outer model of V , and that there exists a branch B through U such that $\sup(B) = \infty = W \cap \text{OR}$ and*

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$(W; V, B)$ satisfies ZFC. Then there exists a $(V; B)$ -definable partial ordering $\mathbb{P}$ and a filter G on $\mathbb{P}$ such that G is generic over $(V; \mathbb{P})$ and $W = V[G]$.

It follows that, for example, *any* countable model W of $\text{ZFC} + “0^\# \text{ exists}”$ is a generic extension of L^W (see Corollary 5.3). It was already known [S2] that *there exist* generic outer models of L^W satisfying “ $0^\#$ exists”.

Remark 1.3. In Theorem 1.2, generally G is generic only with respect to the language of set theory: Let φ be a sentence in the language with equality and a binary relation symbol for the membership relation, as well as a constant symbol for each forcing term. Then $V[G]$ satisfies φ iff there exists a condition in G that forces φ . A condition p forces φ if $V[G']$ satisfies φ , for comeagerly many G' from the neighborhood $\{G' : p \in G'\}$ in the Stone space of $\mathbb{P}$.

Genericity in this sense is, in general, stronger than meeting all predense sets that are elements of V (“internal genericity” in this paper), and weaker than meeting all dense classes that are definable over $(V; \mathbb{P})$ (“definable genericity” in this paper). These three notions collapse to one in the case of set forcing.

The point of this paper is not to develop some esoteric theory of abstract forcing. Nor is it to argue that our definition of “generic” is universally “right”. Instead, it is to show that there is little, if any, room for a general method for constructing outer models other than forcing. This requires both a theorem and a thesis. The thesis is that Theorem 1.2 (and Theorem 1.5 below) do not simply turn on a quirky definition of “generic”. Sections 1.2 and 2 address this.

Remark 1.4. In Theorem 1.2, $(V[G]; G)$ is not a model of ZFC in a language with a predicate symbol for the generic filter G . Predicates for V and $\mathbb{P}$ can be allowed. This is a phenomenon that occurs sometimes in class forcing, but not in set forcing. For example, stationary tower extensions [W] in general do not satisfy ZFC in a language with a predicate symbol for the generic filter.

To some degree, this is an inessential feature of Theorem 1.2. Hugh Woodin observed that, with a stronger non-minimality hypothesis, Theorem 1.2 can be improved to get $(V[G]; V, \mathbb{P}, G) \models \text{ZFC}$. The following theorem is presented in Section 6 of this paper.

Theorem 1.5. (Woodin) *Assume that W is a countable standard transitive outer model of V and that $\infty = \text{OR} \cap W$ is definably regular in $\text{Hyp}(W; V)$. There exists a partial ordering $\mathbb{P}$ and a filter G such that*

- $\mathbb{P}$ is V -amenable and definable over $\text{Hyp}(V)$;
- G is generic over $(V; \mathbb{P})$;
- $W = V[G]$; and
- $(V[G]; V, \mathbb{P}, G) \models \text{ZFC}$.

In general, it is still only with respect to the language of set theory that G is generic, not with respect to a language with auxiliary predicate symbols for these relations. This is necessary:

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Remark 1.6. Assume that $V = L^W = L_\infty$ and W satisfies “ $0^\#$ exists”. Suppose that $\mathbb{P}$ is an L_∞ -amenable partial ordering and let G be generic with respect to the language of set theory augmented by predicate symbols for $\mathbb{P}$ and its ordering. If $W = L_\infty[G]$, then $(L_\infty[G]; \mathbb{P}, G) \not\models \text{ZFC}$. (Otherwise, satisfaction for $(L_\infty; \mathbb{P})$ would be definable from the forcing relation in $(L_\infty; \mathbb{P})$ using that a tail of the Silver indiscernibles are indiscernible in $(L_\infty; \mathbb{P})$. See [F2] or [S2] for details.)

In fact, it is $\mathbb{P}$, rather than G that is the root of the problem. Provided that the forcing relation restricted to sentences of bounded complexity is definable over $(L^W; \mathbb{P})$ (see Remark 1.8 regarding this hypothesis), we have $(L_\infty[G]; \mathbb{P}) \not\models \text{ZFC}$.

Note that this observation does not limit *how generic* G is. It prescribes a feature of $(V[G]; \mathbb{P})$ when G is sufficiently generic and $V[G]$ satisfies “ $0^\#$ exists” and $V = L^V$.

Remark 1.7. In the statement of Theorem 1.2, note the hypothesis that $(W; V, B)$ satisfies ZFC. If V is a canonical inner model (eg., V is L^W) and B is definable over W , then this is equivalent to W being an outer model of V . However, it is possible to have an outer model W of a model V such that $(W; V)$ is not a model of ZFC.

1.2. Class genericity

Why are there so many types of genericity in class forcing?

Beller, Jensen, and Welch [BJW] speak of “strong”, “medium”, and “weak” genericity. Sy Friedman [F2] uses “generic” in a sense close to “medium” in their sense, though not specialized to L . He also uses the terminology “literally generic”. The author is not blameless, with “invisibly” and “hyper” [S2], as well as “amenably” and “weakly” [S1]. (The subscripts “nwg” and “nag” used above stand for “not weakly generic” and “not amenably generic”, respectively.)

Before beginning to answer this question, it should be emphasized that all of our work will be cast in terms of standard transitive *set* models. Granted that our metatheory is ZFC, this is the only reliable way to approach the question with which we began—*is there any general method for constructing outer models other than forcing?* Working in some class theory would risk begging the question.

Modifiers of “generic” address the following three questions:

- (1) How definable is $\mathbb{P}$ with respect to V ?
 - (a) $\mathbb{P}$ is an element of V , or
 - (b) $\mathbb{P}$ is a definable class over V , or
 - (c) $\mathbb{P}$ is a class amenable to V , or
 - (d) $\mathbb{P}$ is a class amenable to some outer model V' of V .
- (2) Which dense classes is the maximal filter G required to meet?
 - (a) G meets every predense $d \subseteq \mathbb{P}$ that is an element of V . For current purposes, let us call such a G **internally generic over $(V; \mathbb{P})$** .
 - (b) G meets $\{p \in \mathbb{P} : p \Vdash \varphi \text{ or } p \Vdash \neg\varphi\}$, for all sentences φ of the forcing language. This is equivalent to G is **generic over $(V; \mathbb{P})$** in the terminology of this paper, cf., Section 2.1.
 - (c) G meets every dense class $D \subseteq \mathbb{P}$ that is definable (perhaps with parameters) over $(V; \mathbb{P})$. For current purposes, let us call such a G **definably generic over $(V; \mathbb{P})$** .

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- (d) G meets every dense class $D \subseteq \mathbb{P}$ that is definable (perhaps with parameters) over $\text{Hyp}(V; \mathbb{P})$, the smallest admissible set with $(V; \mathbb{P})$ as an element. For current purposes let us call such a G *hyper-generic over $(V; \mathbb{P})$* .
- (e) And so forth.
- (3) What special properties does $V[G]$, perhaps with auxiliary predicates, have relative to V ?¹

If $\mathbb{P} \in V$ almost all of these notions are equivalent. (Hyper-genericity is an exception if V is not sufficiently non-minimal.) In contrast, class forcing in any general sense resists formalization in set theory, so it is not surprising that distinctions that are not strictly set-theoretic are important.

Nevertheless, the meaning of “generic” is not up for grabs. Instead, the locutions “ G is generic over $(V; \mathbb{P})$ ”, “ G is $\mathbb{P}$ -generic over V ”, and so forth, repress a parameter, namely, the collection of sentences with respect to which G is generic, or, perhaps, the family of meager sets that G is required to avoid.

Remark 1.8. In set forcing, the forcing relation $\Vdash$ is always definable when restricted to sentences of bounded logical complexity. This is almost always, but not universally true in class forcing.

On the one hand, if $(V[G]; V, \mathbb{P}, G) \models \text{ZFC}$, whenever G is definably generic over $(V; \mathbb{P})$, then the forcing relation is definable. On the other, there does exist an example, due to Sy Friedman, of a class forcing for which the forcing relation is not definable. In this example, $V[G]$ is not admissible, even in a language without auxiliary predicates. It is known that if $V[G]$ is admissible whenever G is hyper-generic over $(V; \mathbb{P})$, then the forcing relation restricted to sentences of bounded complexity is definable in $(V; \mathbb{P})$. However, I am not comfortable asking this result to bear much weight, because the proof uses that $V[G]$ is defined using Shoenfield terms, rather than, say, ramified terms. (This makes Σ_0 -separation in $V[G]$ non-trivial.)

The forcing relation is definable in all of the forcings used in this paper.

1.3. Remarks on the proof

In a way, the proof of any soft theorem like Theorem 1.2 or 1.5 has to be disappointing. Inevitably, it resembles the Completeness Theorem. There are, however, several challenges to meet. It might be useful to discuss these informally.

The proof of Theorem 1.5 is similar to that of Theorem 1.2, so let us concentrate for the moment on Theorem 1.2.

Conditions in $\mathbb{P}$ work to build a Henkin style model $\widehat{V}_G$ (with universe V_G). In this way $\mathbb{P}$ resembles Robinson’s model theoretic forcing. However $\mathbb{P}$ is set theoretic forcing. There are two main tasks. The first of these is arranging that $V_G = V[G]$, where $V[G]$ is the set of values $\dot{a}^G$ of Shoenfield terms. The second task is arranging that the given outer model W is of the form $V[G]$, for some generic G .

Tackling the first of these tasks, why should V_G even be an outer model of V , rather than merely an end-extension that perhaps adds sets not ranked by ordinals in V ? It is

¹ A case can be made that incorporating these into modifiers of “generic” is not appropriate, since they do not bear on how G is generic. In set forcing, for example, one would not say “cardinal preservingly generic”. Again, the author is not blameless. Witness “invisibly generic” in [S2].

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an observation going back to Barwise’s Ph.D. thesis [B1] that model theoretic forcing generically omits types that are not “locally realized”—forced by a single formula to be realized. (For early examples, see [Ks] or, germane to the present context, [HF].)

The type in question is “ x is a set of rank greater than α , for all $\alpha \in V$ ”. This is where the non-minimality hypothesis comes in. The model theoretic aspect of the forcing $\mathbb{P}$ is defined using a theory T^* that locally omits this type. T^* is obtained as a fixed point of an inductive definition. In the case of Theorem 1.2, this fixed point is definable over $(V; B)$, where B is the branch through the tree U mentioned in its statement.

Once we have arranged that V_G is an outer model of V , it is easy to get $V_G \subseteq V[G]$.

The converse inclusion $V[G] \subseteq V_G$ also requires attention. The challenge is to define the forcing in such a way that there are not too many Shoenfield terms. Roughly, the idea is to define $\mathbb{P}$ in such a way that each element of $V[G]$ is named by a Shoenfield term that depends only on sentences of bounded complexity in the model theoretic aspect of the forcing, for which satisfaction is definable.

Our second task is arranging that the given outer model W is $V[G]$, for some generic G . This might seem improbable. Were we just doing model theoretic forcing over V , certainly there would exist non-generic end-extensions V_G . The key fact is that W not an arbitrary end-extension of V ; it is an outer model.

We define a translation procedure $\varphi \mapsto \varphi^*$ from sentences in a large enough fragment of the forcing language into the infinitary language of $\mathbb{P}$ ’s model theoretic aspect and show (Theorem 4.4) that any filter that decides all of the φ^* s is generic. W has an expansion $\widehat{W}$ that decides all of the φ^* s (see the proof of Theorem 5.2). This gives rise to a filter deciding all of the φ^* s (Proposition 4.1), hence a generic filter G such that $W = V_G = V[G]$.

Arranging this requires some care. We need (Lemma 3.1) that the satisfaction relation for quantifier-free φ^* s is definable over $\widehat{W}$. This accounts for two technical points. In order that the infinitary language of the φ^* s be definable in $\widehat{W}$, we use the hypothesis that $(W; V)$ satisfies ZFC. To get quantifier-free satisfaction defined uniformly, a little care is needed in our choice of Henkin terms. Rather than constant symbols, we use terms of the form $F(\underline{\alpha})$, where F is a (single fixed) function symbol and $\underline{\alpha}$ is a constant symbol required by some background axioms to denote the ordinal α .

2. Genericity

In this section, we characterize genericity and compare it to internal genericity, definable genericity, and hyper-genericity. In summary,

- (1) “Generic” means generic with respect to the language of set theory.
- (2) *Definably generic* is equivalent to genericity with respect to an expanded language that has predicate symbols for class Shoenfield terms that are definable over $(V; \mathbb{P})$, provided the usual forcing relation is definable over $(V; \mathbb{P})$ when restricted to sentences of bounded complexity. A “class Shoenfield term” is a $\mathbb{P}$ -value class of ordinary (set) Shoenfield terms, cf., Section 2.2.
- (3) *Hyper-generic* is equivalent to genericity with respect to an expanded language that has predicate symbols for class Shoenfield terms that are definable over $\text{Hyp}(V; \mathbb{P})$.

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(4) It follows that

$$\text{hyper-generic} \Rightarrow \text{definably generic} \Rightarrow \text{generic} \Rightarrow \text{internally generic}.$$

In general, none of these implications can be reversed.

- (5) Two examples illustrate that, in general, internal genericity is not enough and definable genericity is more than enough for genericity.
 - (a) Any countable standard transitive model of ZFC is an internally generic extension of its L , but is never a generic extension via this forcing.
 - (b) If $\mathbb{P}$ is the Easton support product that adds a Cohen subset to each successor cardinal and ω , then any internally generic G is generic. However, if $\infty = V \cap \text{OR}$ is not “definably Mahlo”, then there exist generic G ’s that are not definably generic.

Much of the work in this section is routine. The reader whose primary interest is in the main theorems may safely skip to Section 3.

2.1. Genericity in the language of set theory

We begin by defining a *semantic forcing relation* $\Vdash$ between conditions and sentences of the forcing language: $p \Vdash \varphi$ iff $V[G] \models \varphi$, for almost every maximal filter G on $\mathbb{P}$ with $p \in G$. We say that a filter G on $\mathbb{P}$ is *semantically generic* if, for all sentences φ of the forcing language, $V[G] \models \varphi$ iff $p \Vdash \varphi$, for some condition $p \in G$.

Natural though this relation may be, its definition is not first-order over $(V; \mathbb{P})$.

Next we define a *syntactic forcing relation* $\Vdash$ and isolate exactly the dense subclasses that a filter G on $\mathbb{P}$ must meet in order that $V[G] \models \varphi$ iff $p \Vdash \varphi$, for some $p \in G$. The relation $\Vdash$, when restricted to formulas of bounded complexity, is sometimes definable over $(V; \mathbb{P})$. It is always definable over the least admissible set containing $(V; \mathbb{P})$.

A calculation shows that the two relations $\Vdash$ and $\Vdash$ coincide. Hence semantic and syntactic genericity coincide in, simply, genericity.

Before getting started, let us be clear about the forcing language.

The forcing language

Suppose that $\mathbb{P}$ is a V -amenable partial ordering and that $(V; \mathbb{P})$ is a standard countable transitive model of ZFC in a language with predicate symbols for $\mathbb{P}$ ’s underlying field and its ordering. The class of ***Shoenfield terms*** is defined as usual by recursion:

$$\dot{a} \in V^{\mathbb{P}} \quad \text{iff} \quad \dot{a} \subseteq V^{\mathbb{P}} \times \mathbb{P} \text{ and } \dot{a} \in V.$$

If $\dot{a}$ is a Shoenfield term, let

$$\begin{aligned} \text{dom}(\dot{a}) &= \left\{ \dot{c} : (\dot{c}, p) \in \dot{a}, \text{ for some } p \right\} \text{ and} \\ \dot{a}(\dot{c}) &= \left\{ p : (\dot{c}, p) \in \dot{a} \right\}. \end{aligned}$$

The ***forcing language*** is the first-order language without equality having two binary relation symbols $\hat{\in}$ and $\hat{\subseteq}$ and a constant symbol for each Shoenfield term. We identify Shoenfield terms with their names in the forcing language. Because we are

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concerned only with transitive standard models, which automatically satisfy extensionality, we can introduce equality as a defined relation using $\hat{=}$:

$$t \hat{=} t' \quad \text{abbreviates} \quad (t \hat{\subseteq} t' \wedge t' \hat{\subseteq} t).$$

$G \subseteq \mathbb{P}$ is a **filter** on $\mathbb{P}$ if (1) any condition extended by some element of G is itself an element of G , and (2) any two conditions in G have a common extension that lies in G .

If G is a filter on $\mathbb{P}$, define the **value** $\dot{a}^G$ of the Shoenfield term $\dot{a}$ by

$$\dot{a}^G = \left\{ \dot{c}^G : \dot{a}(\dot{c}) \cap G \neq \emptyset \right\}$$

and set

$$V[G] = \left\{ \dot{a}^G : \dot{a} \in V^{\mathbb{P}} \right\}.$$

If G is non-empty, then $V \subseteq V[G]$ and $V[G]$ is a standard transitive model of Extensionality, Foundation, Pairing, Union, and Infinity. If Collection and Separation hold in $V[G]$, then it is a model of ZFC.

A key feature of forcing languages is that every element of $V[G]$ is named by a constant symbol in the forcing language.

We write “ $V[G] \models \varphi$,” where φ is a sentence of the forcing language, for the cumbersome but more explicit “ $(V[G]; \in, \subseteq, \dot{a}^G)_{\dot{a} \in V^{\mathbb{P}}} \models \varphi$.” A caveat is that in general this is not a definable expansion of $(V[G]; \in, \subseteq)$ because V , $\mathbb{P}$, or G may fail to be definable in this structure. However, set sized fragments of this expansion are definable over $(V[G], \in)$, because $\mathbb{P}$, V and G are $V[G]$ -amenable.

Semantic forcing

Let $\text{StSpc}(\mathbb{P})$ denote the Stone Space of the partial ordering $\mathbb{P}$, that is, the collection of all maximal (under $\subseteq$) filters on $\mathbb{P}$. If $X \subseteq \mathbb{P}$, let

$$\llbracket X \rrbracket = \left\{ G \in \text{StSpc}(\mathbb{P}) : G \cap X \neq \emptyset \right\}.$$

Then $\llbracket X \rrbracket$ is an open set in $\text{StSpc}(\mathbb{P})$. If $p \in \mathbb{P}$, let us write “ $\llbracket p \rrbracket$ ” for the fussier “ $\llbracket \{p\} \rrbracket$.”

If φ is a sentence of the forcing language, define

$$\text{Mod}(\varphi) = \left\{ G \in \text{StSpc}(\mathbb{P}) : V[G] \models \varphi \right\}.$$

Proposition 2.1. *Mod(φ) is a Borel, hence Baire, subset of $\text{StSpc}(\mathbb{P})$.*

PROOF: We define evidently Borel sets $^*\text{Mod}(\varphi)$ and show that $\text{Mod}(\varphi) = ^*\text{Mod}(\varphi)$.

Define $^*\text{Mod}(\varphi)$ by recursion on sentences φ of the forcing language:

$$\begin{aligned} ^*\text{Mod}(\dot{a} \hat{=} \dot{b}) &= \bigcup_{\dot{c} \in \text{dom}(\dot{b})} \left(\llbracket \dot{b}(\dot{c}) \rrbracket \cap ^*\text{Mod}(\dot{a} \hat{\subseteq} \dot{c}) \cap ^*\text{Mod}(\dot{c} \hat{\subseteq} \dot{a}) \right) \\ ^*\text{Mod}(\dot{b} \hat{\subseteq} \dot{a}) &= \bigcap_{\dot{c} \in \text{dom}(\dot{b})} \left((\text{StSpc}(\mathbb{P}) \setminus \llbracket \dot{b}(\dot{c}) \rrbracket) \cup ^*\text{Mod}(\dot{c} \hat{=} \dot{a}) \right) \\ ^*\text{Mod}(\varphi \wedge \psi) &= ^*\text{Mod}(\varphi) \cap ^*\text{Mod}(\psi) \\ ^*\text{Mod}(\neg \varphi) &= \text{StSpc}(\mathbb{P}) \setminus ^*\text{Mod}(\varphi) \\ ^*\text{Mod}(\forall x \varphi) &= \bigcap_{\dot{c} \in V^{\mathbb{P}}} ^*\text{Mod}(\varphi(\dot{c}/x)) \end{aligned}$$

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Now a straightforward induction on φ shows that ${}^*\text{Mod}(\varphi) = \text{Mod}(\varphi)$. Let us consider one representative case:

$$\begin{aligned}
G \in \text{Mod}(\dot{b} \hat{=} \dot{a}) & \text{ iff } V[G] \models \dot{b} \hat{=} \dot{a} \\
& \text{ iff } \dot{c}^G \in \dot{b}^G \Rightarrow \dot{c}^G \in \dot{a}^G, \text{ for all } \dot{c} \in \text{dom}(\dot{b}) \\
& \text{ iff } \dot{b}(\dot{c}) \cap G \neq \emptyset \Rightarrow \dot{c}^G \in \dot{a}^G, \text{ for all } \dot{c} \in \text{dom}(\dot{b}) \\
& \text{ iff } G \notin \llbracket \dot{b}(\dot{c}) \rrbracket \text{ or } G \in {}^*\text{Mod}(\dot{c} \hat{=} \dot{a}), \text{ for all } \dot{c} \in \text{dom}(\dot{b}) \\
& \text{ iff } G \in \bigcap_{\dot{c} \in \text{dom}(\dot{b})} \left((\text{StSpc}(\mathbb{P}) \setminus \llbracket \dot{b}(\dot{c}) \rrbracket) \cup {}^*\text{Mod}(\dot{c} \hat{=} \dot{a}) \right) \\
& \text{ iff } G \in {}^*\text{Mod}(\dot{b} \hat{=} \dot{a}) \quad \square
\end{aligned}$$

Define the **semantic forcing relation** by

$$p \Vdash \varphi \quad \text{iff} \quad \text{Mod}(\varphi) \cap \llbracket p \rrbracket \text{ is comeager in } \llbracket p \rrbracket.$$

Say that a filter G on $\mathbb{P}$ is **semantically generic** over $(V; \mathbb{P})$ if

$$V[G] \models \varphi \quad \text{iff} \quad \exists p \in G \, p \Vdash \varphi,$$

for all sentences φ of the forcing language.

Thus, true to its name, a model $V[G]$ is semantically generic if everything that is true in it is true almost everywhere in an open neighborhood of it.

Syntactic forcing

If $X \subseteq \mathbb{P}$, set

$$[X] = \left\{ q : q \leq p, \text{ for some } p \in X \right\}.$$

Again, we write “ $[p]$ ” for $[\{p\}]$.”

We define certain classes of conditions $F^+(\varphi)$ and $F^-(\varphi)$ by recursion on sentences φ of the forcing language. The idea is that if G meets enough dense subclasses of $\mathbb{P}$, then conditions in $F^+(\varphi)$ make φ true in $V[G]$ and conditions in $F^-(\varphi)$ make φ false. Indeed, conditions in $F^+(\varphi)$ “strongly force” φ in the traditional sense.

In general the recursive definition of $F^\pm(\varphi)$ cannot be carried out inside V , though it can be carried out inside any admissible set containing $(V; \mathbb{P})$.

In each case of the recursion we shall define either $F^+(\varphi)$ or $F^-(\varphi)$. The other class is defined by

$$F^\mp(\varphi) = \left\{ q : F^\pm(\varphi) \cap [q] = \emptyset \right\}.$$

Set

$$F(\varphi) = F^+(\varphi) \cup F^-(\varphi).$$

It is evident, then, that conditions in $F^+(\varphi)$ and $F^-(\varphi)$ are incompatible and that $F(\varphi)$ is dense in $\mathbb{P}$.

Define

$$\begin{aligned}
 F^+(\dot{a} \hat{=} \dot{b}) &= \bigcup_{\dot{c} \in \text{dom}(\dot{b})} \left([\dot{b}(\dot{c})] \cap F^+(\dot{a} \hat{=} \dot{c}) \cap F^+(\dot{c} \hat{=} \dot{a}) \right) \\
 F^-(\dot{b} \hat{=} \dot{a}) &= \bigcup_{\dot{c} \in \text{dom}(\dot{b})} \left([\dot{b}(\dot{c})] \cap F^-(\dot{c} \hat{=} \dot{a}) \right) \\
 F^+(\varphi \wedge \psi) &= F^+(\varphi) \cap F^+(\psi) \\
 F^+(\neg \varphi) &= F^-(\varphi) \\
 F^-(\forall x \varphi) &= \bigcup_{\dot{c} \in V^{\mathbb{P}}} F^-(\varphi(\dot{c}/x))
 \end{aligned}$$

Note that $F^+(\varphi)$ and $F^-(\varphi)$ are both open subsets of $\mathbb{P}$.

Define the **syntactic forcing relation** by

$$p \Vdash \varphi \quad \text{iff} \quad F^+(\varphi) \text{ is dense below } p.$$

Say that a filter G on $\mathbb{P}$ is **syntactically generic** over $(V; \mathbb{P})$ if

$$V[G] \models \varphi \quad \text{iff} \quad \exists p \in G \, p \Vdash \varphi,$$

for all sentences φ of the forcing language.

It follows from the next lemma that syntactically generic filters are comeager in $\text{StSpc}(\mathbb{P})$.

Lemma 2.2. *If $G \cap F(\varphi) \neq \emptyset$, for all sentences φ of the forcing language, then G is syntactically generic.*

PROOF: Suppose that G meets $F(\varphi) = F^+(\varphi) \cup F^-(\varphi)$, for all sentences φ of the forcing language, and proceed by induction on φ to see that $V[G] \models \varphi$ iff G meets $F^+(\varphi)$.

If φ is $\dot{a} \hat{=} \dot{b}$, note that the following statements are equivalent:

- $V[G] \models \dot{a} \hat{=} \dot{b}$
- $\exists \dot{c} \in \text{dom}(\dot{b}) \, (\dot{a}^G = \dot{c}^G \text{ and } G \cap \dot{b}(\dot{c}) \neq \emptyset)$
- $\exists \dot{c} \in \text{dom}(\dot{b}) \, \exists p_1, p_2, p_3 \in G$
 $(p_1 \text{ meets } F^+(\dot{a} \hat{=} \dot{c}) \text{ and } p_2 \text{ meets } F^+(\dot{c} \hat{=} \dot{a}) \text{ and } p_3 \in \dot{b}(\dot{c}))$
- $\exists p \in G \, \exists \dot{c} \in \text{dom}(\dot{b}) \, (p \text{ meets } F^+(\dot{a} \hat{=} \dot{c}) \text{ and } F^+(\dot{c} \hat{=} \dot{a}) \text{ and } \dot{b}(\dot{c}))$
- $p \text{ meets } F^+(\dot{a} \hat{=} \dot{b})$, for some $p \in G$

If φ is $\dot{b} \hat{=} \dot{a}$, note that the following statements are equivalent:

- $V[G] \models \dot{b} \hat{=} \dot{a}$
- $\forall \dot{c} \in \text{dom}(\dot{b}) \, (G \cap \dot{b}(\dot{c}) \neq \emptyset \Rightarrow G \cap F^+(\dot{c} \hat{=} \dot{a}) \neq \emptyset)$
- $\forall \dot{c} \in \text{dom}(\dot{b}) \, (G \cap \dot{b}(\dot{c}) \neq \emptyset \Rightarrow G \cap F^-(\dot{c} \hat{=} \dot{a}) = \emptyset)$
- $G \cap F^-(\dot{b} \hat{=} \dot{a}) = \emptyset$

[Note the contrapositive of each of $(\Leftarrow)$ and $(\Rightarrow)$.]

- $G \cap F^+(\dot{b} \hat{=} \dot{a}) \neq \emptyset$

The cases when φ is either of the form $\psi \wedge \theta$ or of the form $\neg\psi$ are easy by induction.

If φ is $\forall x\psi$, note that the following statements are equivalent:

- $V[G] \models \forall x\psi$
- $V[G] \models \psi(\dot{c}/x)$, for all Shoenfield terms $\dot{c}$
- $G \cap F^+(\psi(\dot{c}/x)) \neq \emptyset$, for all $\dot{c}$
- $G \cap F^-(\psi(\dot{c}/x)) = \emptyset$, for all $\dot{c}$
- $G \cap F^+(\forall x\psi) \neq \emptyset$

For the last ($\Rightarrow$) implication, note that if $G \cap F^+(\forall x\psi) = \emptyset$, then $G \cap F^-(\forall x\psi) \neq \emptyset$. So $G \cap F^-(\psi(\dot{c}/x)) \neq \emptyset$, for some $\dot{c}$.

To see the final ($\Leftarrow$) implication, note that if $G \cap F^-(\psi(\dot{c}/x)) \neq \emptyset$, for some $\dot{c}$, then $G \cap F^-(\forall x\psi) \neq \emptyset$, so $G \cap F^+(\forall x\psi) = \emptyset$. $\square$

Proposition 2.3. $p \Vdash \varphi$ iff $p \Vdash \varphi$

PROOF: Suppose first that $p \Vdash \varphi$. If $G \in \text{StSpc}(\mathbb{P})$ is syntactically generic and $p \in G$, then $V[G] \models \varphi$. Since syntactically generic G are comeager, it follows that $\text{Mod}(\varphi)$ is comeager in $\llbracket p \rrbracket$. Hence $p \Vdash \varphi$.

Conversely, if $p \nVdash \varphi$, then there exists $q \leq p$ meeting $F^-(\varphi)$. If $G \in \text{StSpc}(\mathbb{P})$ is syntactically generic and $q \in G$, then $V[G] \nVdash \varphi$. Hence $\text{Mod}(\varphi)$ is not comeager in $\llbracket p \rrbracket$. $\square$

Characterization Lemma 2.4. Suppose that $G \in \text{StSpc}(\mathbb{P})$. The following three statements are equivalent:

- G is semantically generic.
- G is syntactically generic.
- $G \cap F(\varphi) \neq \emptyset$, for all sentences φ of the forcing language $\mathcal{L}$.

If any of these equivalent statements hold, let us say that G is (simply) **generic** over $(V; \mathbb{P})$.

PROOF: The only matter requiring attention is seeing that if G is syntactically generic, then G meets $F(\varphi)$, for all sentences φ . Assume that G is syntactically generic. Proceed by induction on φ simultaneously to argue that $V[G] \models \varphi$ iff $G \cap F^+(\varphi) \neq \emptyset$ and $V[G] \nVdash \varphi$ iff $G \cap F^-(\varphi) \neq \emptyset$. In each case, first establish this for whichever of $F^+(\varphi)$ and $F^-(\varphi)$ is explicitly defined. Then establish this claim for the other, using that if conditions with no extension in $F^\pm(\varphi)$ are dense below p , then p has no extension in $F^\pm(\varphi)$, hence itself lies in $F^\mp(\varphi)$. $\square$

Remark 2.5. Of course, the author makes no claim originality. With respect to set forcing, this account is completely standard (see, for example, [Kc]). Indeed, the “generic set” Cohen constructs in [C] is given by a “complete sequence” $\langle P_n : n \in \omega \rangle$, where the n^{th} sentence of the forcing language holds or fails comeagerly often in the neighborhood $\{G' : P_n \in G'\}$.

Remark 2.6. A filter $G \in \text{StSpc}(\mathbb{P})$ is generic in any of the following four cases:

- (1) G is internally generic, that is, meets every predense class that lies in V , and, given any $(V; \mathbb{P})$ -definable dense D , there exists $\kappa < \infty$ such that $D \cap V_\kappa$ is predense. For example, this happens if $\mathbb{P} \in V$.
- (2) The forcing relation $\Vdash$, when restricted to sentences of bounded complexity, is definable over $(V; \mathbb{P})$ and G is definably generic over $(V; \mathbb{P})$, that is, G meets every $(V; \mathbb{P})$ -definable dense subclass of $\mathbb{P}$.
- (3) G meets every $(V; \mathbb{P}, \Vdash_a)$ -definable dense subclass of $\mathbb{P}$, where $\mathbb{P}$ is V -amenable and $\Vdash_a$ is the forcing relation restricted to atomic sentences.
- (4) G is hyper-generic over $(V; \mathbb{P})$, that is, meets every dense subset of $\mathbb{P}$ that is definable over the least admissible set with $(V; \mathbb{P})$ as an element.

2.2. Genericity in larger languages

In this section we generalize genericity to expansions of the language of set theory and use this generalization to characterize definable genericity and hyper-genericity.

Let $(V; \mathbb{P})$ be a countable standard transitive model of ZFC. A **class Shoenfield term** is a V -amenable $\dot{A} \subseteq V^{\mathbb{P}} \times \mathbb{P}$. If $\dot{b}$ is an ordinary set Shoenfield term, let $\dot{A}(\dot{b}) = \{p : (\dot{b}, p) \in \dot{A}\}$. If G is a filter on $\mathbb{P}$, the value given $\dot{A}$ by G is $\dot{A}^G = \{\dot{b}^G : \dot{A}(\dot{b}) \cap G \neq \emptyset\}$.

Suppose that $\mathcal{F}$ is a non-empty family of V -amenable subsets of V such that if $A \in \mathcal{F}$ and B is definable (perhaps with parameters) over $(V; A)$, then $B \in \mathcal{F}$.

There are two examples relevant to the current discussion. To characterize definable genericity, $\mathcal{F}$ will be the collection of subsets of V that are definable over $(V; \mathbb{P})$. To characterize hyper-genericity, $\mathcal{F}$ will be the collection of subsets of V that are definable over $\text{Hyp}(V; \mathbb{P})$.

Say that a filter G on $\mathbb{P}$ is **$\mathcal{F}$ -generic** over $(V; \mathbb{P})$ if it meets every dense subset of $\mathbb{P}$ that lies in $\mathcal{F}$.

Let $\mathcal{L}(\mathcal{F})$ be the first-order language that adds to the usual set forcing language, $\mathcal{L} = \{\dot{\in}, \dot{\subseteq}\} \cup \{\dot{a} : \dot{a} \text{ is a set Shoenfield term}\}$, a one-place predicate symbol for each class Shoenfield term in $\mathcal{F}$. If G is a filter on $\mathbb{P}$, let $V[G]^\mathcal{F}$ be the expansion of $V[G]$ to an $\mathcal{L}(\mathcal{F})$ -structure obtained by interpreting the predicate symbol denoting $\dot{A}$ by $\dot{A}^G$.

The semantic and syntactic forcing relations are easily extended to $\mathcal{L}(\mathcal{F})$ -sentences. Declare that

$$p \Vdash^\mathcal{F} \varphi \quad \text{iff} \quad \left\{ G \in \text{StSpc}(\mathbb{P}) : V[G]^\mathcal{F} \models \varphi \right\} \cap [p] \text{ is comeager in } [p].$$

Say that G is *semantically generic with respect to $\mathcal{L}(\mathcal{F})$* when

$$V[G]^\mathcal{F} \models \varphi \quad \text{iff} \quad \exists p \in G \ p \Vdash^\mathcal{F} \varphi,$$

for all sentences φ of $\mathcal{L}(\mathcal{F})$.

Let the syntactic forcing relation $\Vdash^\mathcal{F}$ be defined like $\Vdash$, adding to the definition of $F^\pm(\varphi)$ the clause

$$F^+(\dot{A}\dot{b}) = \bigcup_{\dot{c} \in V^\mathbb{P}} [\dot{A}(\dot{c})] \cap F^+(\dot{b} \dot{\subseteq} \dot{c}) \cap F^+(\dot{c} \dot{\subseteq} \dot{b}).$$

2.2. GENERICITY IN LARGER LANGUAGES

Say that G is *syntactically generic with respect to $\mathcal{L}(\mathcal{F})$* when

$$V[G]^\mathcal{F} \models \varphi \quad \text{iff} \quad \exists p \in G \ p \Vdash^\mathcal{F} \varphi,$$

for all sentence φ of $\mathcal{L}(\mathcal{F})$.

It is straightforward to extend the proof of Lemmas 2.2 and 2.4 to get

Characterization Lemma 2.7. *Suppose that $G \in \text{StSpc}(\mathbb{P})$. Then the following three statements are equivalent:*

- G is *semantically generic with respect to $\mathcal{L}(\mathcal{F})$* .
- G is *syntactically generic with respect to $\mathcal{L}(\mathcal{F})$* .
- $G \cap F(\varphi) \neq \emptyset$, for all sentences φ of the language $\mathcal{L}(\mathcal{F})$.

If any of these equivalent statements hold, say that G is **generic over $(V; \mathbb{P})$ with respect to $\mathcal{L}(\mathcal{F})$** .

Let $\Vdash_{\text{qf}}^{\dot{A}_1, \dots, \dot{A}_n}$ be the forcing relation restricted to quantifier-free sentences of $\mathcal{L}(\mathcal{F})$ in which only $\dot{A}_1, \dots, \dot{A}_n$ occur.

Proposition 2.8. *Suppose that $\Vdash_{\text{qf}}^{\dot{A}_1, \dots, \dot{A}_n}$ lies in $\mathcal{F}$, for all $\dot{A}_1, \dots, \dot{A}_n \in \mathcal{F}$. Then the following two statements are equivalent:*

- G is $\mathcal{F}$ -generic over $(V; \mathbb{P})$.
- G is generic over $(V; \mathbb{P})$ with respect to $\mathcal{L}(\mathcal{F})$.

PROOF: ($\Rightarrow$) Suppose the class Shoenfield terms occurring in φ are $\dot{A}_1, \dots, \dot{A}_n$. Then $\{p : p \Vdash^\mathcal{F} \varphi \text{ or } p \Vdash^\mathcal{F} \neg \varphi\}$ is definable over $(V; \mathbb{P}, \Vdash_{\text{qf}}^{\dot{A}_1, \dots, \dot{A}_n})$, hence lies in $\mathcal{F}$.

($\Leftarrow$) We may assume that $\mathbb{P}$ has a weakest element $\mathbb{1}$. Suppose that $D \in \mathcal{F}$ is a dense class in $\mathbb{P}$. Set

$$\begin{aligned} \dot{D} &= \{(\check{p}, \mathbb{1}) : p \in D\} \\ \dot{G} &= \{(\check{p}, p) : p \in \mathbb{P}\} \end{aligned}$$

Then $\dot{D}$ and $\dot{G}$ are in $\mathcal{F}$ and

$$\left\{ G' \in \text{StSpc}(\mathbb{P}) : V[G']^\mathcal{F} \models \exists x (\dot{D}x \wedge \dot{G}x) \right\}$$

is comeager in $\text{StSpc}(\mathbb{P})$, since D is dense. It follows that if G is generic with respect to $\mathcal{L}(\mathcal{F})$, then $\emptyset \neq \dot{G}^G \cap \dot{D}^G = G \cap D$. $\square$

Corollary 2.9. *Suppose that G is a filter on $\mathbb{P}$.*

- (1) *Let $\mathcal{F}$ be the family of subsets of V that are definable over $(V; \mathbb{P})$. Assume that the ordinary forcing relation (ie, for $\mathcal{L}$), when restricted to sentences of bounded complexity, lies in $\mathcal{F}$. Then G is definably generic iff G is generic with respect to $\mathcal{L}(\mathcal{F})$.*
- (2) *Let $\mathcal{F}$ be the family of subsets of V that are definable over $\text{Hyp}(V; \mathbb{P})$. Then G is hyper-generic iff G is generic with respect to $\mathcal{L}(\mathcal{F})$.*

PROOF: Note first that if the ordinary forcing relation restricted to quantifier-free sentences, $\Vdash_{\text{qf}}^\emptyset$, lies in $\mathcal{F}$, then $\Vdash_{\text{qf}}^{\dot{A}_1, \dots, \dot{A}_n} \in \mathcal{F}$, for all $\dot{A}_1, \dots, \dot{A}_n \in \mathcal{F}$. This is because the latter relation is definable over $(V; \mathbb{P}, \Vdash_{\text{qf}}^\emptyset, \dot{A}_1, \dots, \dot{A}_n)$.

Then (1) is immediate from the previous proposition.

Even the full forcing relation for $\mathcal{L}$ is definable over $\text{Hyp}(V; \mathbb{P})$. (2) follows. $\square$

2.3. Two examples

In this section we look at two examples. In the first, we see that every model of ZFC is an internally generic extension of its L , but is never a (fully) generic extension via this forcing.

The second example is the Easton support product that adds a Cohen subset to each successor cardinal and ω . In this case, we see that every internally generic G is generic. It follows that if $V \cap \text{OR}$ is not “definably Mahlo”, then there exist generic G that are not definably generic.

2.3.1. Internal genericity is not enough

Recall that G is *internally generic* over $(V; \mathbb{P})$ if G is a maximal filter on $\mathbb{P}$ that meets every predense set that is an element of V .

Example 2.10. Suppose that W is a countable standard transitive model of ZFC. Let “ L ” indicate L^W . There exists an L -definable partial ordering $\mathbb{P}$ satisfying the following:

- (1) There exists a W -amenable filter G that is internally generic over $(L; \mathbb{P})$ and $W = L[G]$ and $(W; \mathbb{P}, G)$ satisfies ZFC.
- (2) If G is a maximal filter on $\mathbb{P}$ and $L[G]$ satisfies ZFC, then G is not generic over $(L; \mathbb{P})$.

PROOF: **For (1)**, set $\infty = \text{OR} \cap W$. If necessary, by forcing over W add a W -amenable sequence $\langle a_\alpha : \alpha < \infty \rangle$ such that $(W; \langle a_\alpha : \alpha < \infty \rangle) \models \text{ZFC}$ and $\langle a_\alpha : \alpha < \infty \rangle$ enumerates W in such a way that if $a_\alpha \in a_\beta$, then $\alpha < \beta$.

Let $\mathcal{K}$ be the language $\{\hat{=}\} \cup \{c_\alpha : \alpha < \infty\}$ (with equality), where $\langle c_\alpha : \alpha < \infty \rangle$ is an L -definable class sequence of constant symbols. Let

$$T = \left\{ \forall x (x \hat{=} c_\alpha \rightarrow \bigvee_{\beta < \alpha} x \hat{=} c_\beta) : \alpha < \infty \right\}.$$

Then T is a consistent L -definable class of axioms in the language $\mathcal{K}$. Indeed, $\widehat{W} = (W; a_\alpha)_{\alpha < \infty}$ satisfies T .

Working in L , define $\mathbb{P}$ to consist of all infinitary quantifier-free sentences p in the admissible language $\mathcal{K}_L$ such that $T \cup \{p\}$ is consistent. That is, $\mathbb{P}$ consists of all infinitary Boolean combinations $p \in L$ of atomic sentences of $\mathcal{K}$ such that $T \cup \{p\}$ is consistent. Let $\mathbb{P}$ be ordered by declaring that $p \geq q$ when $T \vdash q \rightarrow p$. Then $\mathbb{P}$ is definable in L .

Set

$$G = \left\{ p \in \mathbb{P} : \widehat{W} \models p \right\}.$$

Then G is a maximal filter on $\mathbb{P}$ and is definable over $(W; \langle a_\alpha : \alpha < \infty \rangle)$. Hence $(W; \mathbb{P}, G)$ satisfies ZFC. Furthermore, G meets every predense d that lies in L . Indeed, if $d \in L$ is predense in $\mathbb{P}$, then $T \vdash \bigvee d$. It follows that $\widehat{W} \models \bigvee d$, hence $\widehat{W} \models p$, for some $p \in d$. Thus $G \cap d \neq \emptyset$.

Since G is definable over $(W; \langle a_\alpha : \alpha < \infty \rangle)$, we have that $L[G] \subseteq W$. For the converse inclusion, working in L define Shoenfield terms $\dot{a}_\alpha$, for $\alpha < \infty$ by

$$\dot{a}_\alpha = \left\{ (\dot{a}_\beta, \{c_\beta \hat{=} c_\alpha\}) : \beta < \alpha \right\}.$$

Then $\dot{a}_\alpha^G = a_\alpha$, by induction on α . Hence $W \subseteq L[G]$.

For (2), note that, given $\alpha < \infty$ and a condition p , there exists $\beta < \infty$ such that $p \cup \{“c_\beta \text{ is a function from } \omega \text{ onto } c_\alpha”\}$ is a condition. [Choose β larger than α and every γ such that c_γ occurs in p . A little work is required to express the statement in quotes as an infinitary Boolean combination in $\mathcal{K}$. The key point is that “ $\forall x \in c_\alpha \dots$ ” can be expressed by “ $\bigwedge_{\delta < \alpha} (c_\delta \in c_\alpha \rightarrow \dots)$ ”.]

Thus G such that $L[G] \models \text{ZFC}$ are meager in $\text{StSpc}(\mathbb{P})$. $\square$

2.3.2. Definable genericity is more than enough

Recall that a maximal filter G is *definably generic* over $(V; \mathbb{P})$ if G meets every dense subclass of $\mathbb{P}$ that is definable (perhaps with parameters) over $(V; \mathbb{P})$.

Factoring in the restriction imposed in [F2] to models V of the form $L(A)$, where $A \subseteq \text{OR}$, this is what Sy Friedman calls “ $\mathbb{P}$ -generic over V ”.

Recall from §2.2 that definable genericity is equivalent to genericity with respect to the language $\mathcal{L}(\mathcal{F})$ that includes predicate symbols for all V -amenable class Shoenfield terms that are definable over $(V; \mathbb{P})$, at least when the ordinary (set) forcing relation restricted to sentences of bounded complexity is definable over $(V; \mathbb{P})$.

Assume that V is a countable standard transitive model of $\text{ZFC} + \text{GCH}$. Working in V , declare that $p \in \mathbb{P}$ iff p is a (set) function such that $\text{dom}(p)$ is an Easton set of ordinals and $\text{rng}(p) \subseteq \{0, 1\}$. A set of ordinals is Easton if it is bounded below every regular cardinal. Order $\mathbb{P}$ by reverse functional extension.

The forcing $\mathbb{P}$ is equivalent to the Easton support product adding a Cohen subset to every successor cardinal and ω . It is well-known that $\mathbb{P}$ is well-behaved: If G is definably generic over $(V; \mathbb{P})$, then $V[G]$ is an outer model of V that satisfies $\text{ZFC} + \text{GCH}$ and is cofinality preserving. Indeed, $(V[G]; V, G)$ satisfies ZFC in a language augmented by predicate symbols for V and G . Furthermore, $\mathbb{P}$ is “tame”, in the terminology of [F2]. Consequently, the forcing relation is definable over V , when restricted to sentences of bounded complexity.

Set $\infty = V \cap \text{OR}$. Say that ∞ is **definably Mahlo** if every club class of ordinals $C \subseteq \infty$ that is definable (perhaps with parameters) over V contains a V -inaccessible cardinal.

If ∞ is definably Mahlo in V and D is a V -definable dense class, then $D \cap V_\kappa$ is predense, for some $\kappa < \infty$. In this case, it follows that any G that is internally generic is definably generic, hence generic. However, if ∞ is not definably Mahlo, then there do exist internally generic G that are not definably generic:

Example 2.11. Let $\mathbb{P}$ be the forcing defined above that (is equivalent to the forcing that) with an Easton support product adds a Cohen subset to every successor cardinal and ω .

- (1) If G is internally generic over $(V; \mathbb{P})$, then G is generic over $(V; \mathbb{P})$.
- (2) If ∞ is not definably Mahlo, then there exists a filter G that is generic but not definably generic over $(V; \mathbb{P})$.

PROOF: **For (1)**, using Lemma 2.4, it suffices to see that for each sentence φ of the forcing language, there exist sets $f^+(\varphi)$ and $f^-(\varphi)$ in V such that

- (A) $f^+(\varphi) \cup f^-(\varphi)$ is predense in $\mathbb{P}$ and
- (B) $f^\pm(\varphi) \subseteq F^\pm(\varphi)$

2.3.2. DEFINABLE GENERICITY IS MORE THAN ENOUGH

As with the classes $F^\pm(\varphi)$, proceed by induction first on atomic sentences, then on first order sentences, to define one of $f^+(\varphi)$ and $f^-(\varphi)$. The other of the pair is given by

$$p \in f^\mp(\varphi) \quad \text{iff} \quad \text{dom}(p) \subseteq \bigcup \left\{ \text{dom}(q) : q \in f^\pm(\varphi) \right\} \text{ and} \\ \forall q \in f^\pm(\varphi) \exists \alpha \in \text{dom}(p) \cap \text{dom}(q) \left(p(\alpha) \neq q(\alpha) \right).$$

Note that conditions in $f^\mp(\varphi)$ are incompatible with conditions in $f^\pm(\varphi)$ and that $f^+(\varphi) \cup f^-(\varphi)$ is a predense set that lies in V , establishing (A).

For atomic sentences, set

$$f^+(\dot{a} \hat{=} \dot{b}) = \left\{ p \cup q \cup r \in \mathbb{P} : \exists \dot{c} \in \text{dom}(\dot{b}) \left(p \in \dot{b}(\dot{c}) \text{ and} \right. \right. \\ \left. \left. q \in f^+(\dot{a} \hat{=} \dot{c}) \text{ and } r \in f^+(\dot{c} \hat{=} \dot{a}) \right) \right\}$$

and set

$$f^-(\dot{b} \hat{=} \dot{a}) = \left\{ p \cup q \in \mathbb{P} : \exists \dot{c} \in \text{dom}(\dot{b}) \left(p \in \dot{b}(\dot{c}) \text{ and } q \in f^-(\dot{c} \hat{=} \dot{a}) \right) \right\}.$$

Next extend this to quantifier-free sentences by declaring that

$$f^+(\neg\varphi) = f^-(\varphi) \\ f^+(\varphi \wedge \psi) = \left\{ p \cup q \in \mathbb{P} : p \in f^+(\varphi) \text{ and } q \in f^+(\psi) \right\}.$$

A routine induction shows (B) for quantifier-free sentences.

Note that for quantifier-free sentences the function $\varphi \mapsto (f^+(\varphi), f^-(\varphi))$ is definable without parameters over V .

Extending the definitions of $f^\pm(\varphi)$ to quantified sentences takes advantage of $\mathbb{P}$'s homogeneity. Proceed by induction on the complexity of φ , maintaining that

$$(*) \quad \dot{c} \mapsto \left(f^+(\varphi(\dot{c}/x)), f^-(\varphi(\dot{c}/x)) \right)$$

is definable without parameters over V (though the complexity of the definition depends on the complexity of φ).

Given φ , let $n \geq 2$ be least such that $(*)$ is Σ_n -definable without parameters. Let λ be least such that the Shoenfield terms occurring in φ lie in H_λ and $(H_\lambda; \mathbb{P} \cap H_\lambda) \prec_{\Sigma_n} (V; \mathbb{P})$ (so that H_λ is closed under the operation $(*)$). Set

$$f^-(\forall x \varphi) = \bigcup_{\dot{c} \in H_\lambda} f^-(\varphi(\dot{c}/x)).$$

Again, (B) is proved by induction on φ , this time starting with quantifier-free φ , for which (B) is already established. It is clear from its definition (and induction) that $f^-(\forall x \varphi) \subseteq F^-(\forall x \varphi)$. The key point in checking that $f^+(\forall x \varphi) \subseteq F^+(\forall x \varphi)$ is the following

Claim. *If $p \notin F^+(\forall x \varphi)$, then there exists $\dot{c} \in H_\lambda$ such that some condition in $f^-(\varphi(\dot{c}/x))$ is compatible with p .*

PROOF: Let $\kappa < \lambda$ be a cardinal such that the Shoenfield terms occurring in φ lie in H_κ . We may assume that $p \in F^-(\varphi(\dot{c}/x))$, for some $\dot{c}$, hence we may assume that that p meets $f^-(\varphi(\dot{c}/x))$, for some $\dot{c}$, by induction. It follows that

$$H_\lambda \models \exists q \leq (p \restriction \kappa) \exists \dot{c} \left(q \text{ meets } f^-(\varphi(\dot{c}/x)) \right).$$

Let q and $\dot{c}$ be elements of H_λ such that $q \leq p \restriction \kappa$ and q meets $f^-(\varphi(\dot{c}/x))$. Define an automorphism $h: \mathbb{P} \rightarrow \mathbb{P}$ by $h(r) = r'$, where $\text{dom}(r') = \text{dom}(r)$ and

$$r'(\alpha) = \begin{cases} 1 - r(\alpha), & \text{if } \alpha \in (\text{dom}(r) \cap \text{dom}(p) \cap \text{dom}(q)) \setminus \kappa \text{ and } p(\alpha) \neq q(\alpha) \\ r(\alpha), & \text{otherwise.} \end{cases}$$

Let h_* be the induced map on terms,

$$h_*(\dot{a}) = \left\{ (h_*(\dot{b}), h(p)) : (\dot{b}, p) \in \dot{a} \right\}.$$

Then $h_* \restriction H_\kappa = \text{id} \restriction H_\kappa$ and $h(q)$ is compatible with p .

Note that if $\dot{a}_1, \dots, \dot{a}_n$ are the Shoenfield terms occurring in the sentence ψ , then

$$h'' f^\pm(\psi(\dot{a}_1, \dots, \dot{a}_n)) = f^\pm(\psi(h_*(\dot{a}_1), \dots, h_*(\dot{a}_n))).$$

[Proceed by induction on ψ . The only point of any interest is in the quantifier case. Note that the choice of λ is the same when defining $f^-(\forall x \psi(\dot{a}_1, \dots, \dot{a}_n))$ as when defining $f^-(\forall x \psi(h_*(\dot{a}_1), \dots, h_*(\dot{a}_n)))$ because $\dot{a} \in H_\lambda$ iff $h_*(\dot{a}) \in H_\lambda$.]

Set $q' = h(q)$ and $\dot{c}' = h_*(\dot{c})$. Then q' and $\dot{c}'$ lie in H_λ and q' is compatible with p . Finally, q' meets $f^-(\varphi(\dot{c}'/x))$, since q meets $f^-(\varphi(\dot{c}/x))$.

For (2), note that if $H_\lambda \prec_{\Sigma_2} V$ and $d \in H_\lambda$ is predense in $\mathbb{P}$, then H_λ satisfies “ d is predense in $\mathbb{P}$ ”. Using this, in ω steps build an internally generic G avoiding $\{p : p(\alpha) = 1, \text{ for some } \alpha \in C\}$, where $C \subseteq \{\lambda : H_\lambda \prec_{\Sigma_2} V\}$ witnesses that ∞ is not definably Mahlo.

Then G is internally generic, hence generic by part (1), but not definably generic. $\square$

3. The forcing $\mathbb{P}_{T^*}$

In the next three sections, we turn to the proof of Theorem 1.2. In Section 6 this proof is modified to yield Woodin’s Theorem 1.5.

Suppose that $T \in V$ is a set of axioms extending $\text{KP} + \text{Choice}^2$ and is formulated in a finite language extending the language of set theory. In this section we define

² I would like to thank Robert Owen for pointing out that requiring T to extend ZFC is too strong, in general. In Theorem 1.2 we need a T such that $W_\kappa \models T$, for unboundedly many κ , just knowing that W satisfies ZFC.

3. THE FORCING $\mathbb{P}_{T^*}$

a V -amenable class forcing such that, under suitable hypotheses, if V has an outer model satisfying T , then $V[G]$ is an outer model satisfying T when G is generic.

In the next section we shall see, under suitable hypotheses, that *any* outer model of V satisfying T is of the form $V[G]$ for some class G that is generic for such a forcing.

The restrictions that the language of T be finite can be weakened. The underlying requirement is that satisfaction for atomic sentences in this language must be uniformly definable over outer models W of V such that W satisfies T and $(W; V)$ is admissible.

The forcing $\mathbb{P}_{T^*}$ builds a Henkin-style model of T , working in a certain infinitary language consisting of “ V -suitable” formulas. Building a Henkin model requires a language with an adequate supply of closed terms. A little care is needed when introducing these terms because we need that satisfaction of atomic formulas in the enlarged language is definable over the model constructed generically. Working in V , we define an extension $\mathcal{L}^+$ of the language of T that adds these closed terms and define an infinitary extension T^+ of T that enforces properties of these terms, as well as insures that V is contained in any transitive model of T^+ .

An omitting-types requirement must then be imposed so that a generic model is an outer model of V , rather than merely an end-extension. This is done by extending T^+ to a consistent fixed point T^* of a certain positive inductive definition. The axioms T^* are not definable over V in general. The existence of T^* such that $(V; T^*)$ is admissible is an extra hypothesis. It is satisfied if V is sufficiently non-minimal.

The theory T^+

Consider any language $\mathcal{L}$ that is definable over V . The ***V -suitable formulas*** of this language inhabit a certain fragment of the infinitary language $\mathcal{L}_V$. First define the quantifier-free V -suitable formulas as follows:

- Any atomic formula is a quantifier-free V -suitable formula.
- If φ is a quantifier-free V -suitable formula, then so is $(\neg\varphi)$.
- If $\Phi(x_1, \dots, x_k)$ is a set in V of quantifier-free V -suitable formulas, then $\bigvee \Phi$ is a quantifier-free V -suitable formula.

Note that any quantifier-free V -suitable formula mentions only finitely many distinct variables.

General V -suitable formulas constitute the closure of the quantifier-free V -suitable formulas under finitary Boolean connectives and quantifiers:

- If φ is a quantifier-free V -suitable formula, then φ is a V -suitable formula.
- If φ is a V -suitable formula, then so is $(\neg\varphi)$.
- If φ and ψ are V -suitable formulas, then so is $(\varphi \vee \psi)$.
- If φ is a V -suitable formula, then so is $\exists x\varphi$.

The connectives $\wedge$, $\rightarrow$, and $\leftrightarrow$ and the quantifier $\forall$ are introduced as abbreviations in the usual manner.

Now fix a finite language $\mathcal{L}$ with equality ($\hat{=}$) extending the language of set theory $\{\hat{\in}\}$. Fix a countable standard transitive model V of ZFC. Set $\infty = \text{OR} \cap V$ and let $\mathcal{L}^+ = \mathcal{L} \cup \{F, \underline{V}\} \cup \{\underline{a} : a \in V\}$, where

- $\underline{V}$ is a new one-place predicate symbol that will be used to define the inner model V ;

3. THE FORCING $\mathbb{P}_{T^*}$

- $\underline{a}$ is a new constant symbol that will be used to denote a ; and
- F is a new one-place function symbol. The closed terms $F(\underline{\alpha})$, for ordinals $\alpha < \infty$ will serve as Henkin terms.

We assume that sets in V representing the symbols and formulas of $\mathcal{L}^+$ are chosen in some straightforward canonical manner.

Fix a theory $T \in V$ extending $\text{KP} + \text{Choice}$ that is formulated in the finite language $\mathcal{L}$. Let T^+ be the following set of V -suitable sentences in the language $\mathcal{L}^+$:

$$\begin{aligned}
& T \\
& \text{KP in the language } \mathcal{L}^+ \\
& \forall x (x \hat{=} \underline{a} \leftrightarrow \bigvee_{b \in a} x \hat{=} \underline{b}), \text{ for } a \in V \\
& \underline{V} \underline{a}, \text{ for } a \in V \\
& \forall x (\underline{V}x \wedge \text{rk}(x) < \underline{\alpha} \rightarrow \bigvee_{a \in V_\alpha} x \hat{=} \underline{a}), \text{ for } \alpha < \infty \\
& \forall x (x \not\hat{=} \text{OR} \rightarrow F(x) \hat{=} \emptyset) \\
& \forall x (x \hat{=} F(\underline{\alpha}) \rightarrow \bigvee_{\beta < \alpha} x \hat{=} F(\underline{\beta})), \text{ for } \alpha < \infty.
\end{aligned}$$

Note that if there exists an outer model W of V satisfying T such that $(W; \dots, V)$ is admissible, then $(W; \dots, V)$ has an expansion $\widehat{W}$ that is a model of T^+ : Set $\underline{a}^{\widehat{W}} = a$, for all $a \in V$, and set $F^{\widehat{W}}(x) = \emptyset$, for all $x \in W$.

Note also that if $\widehat{W} = (W; \dots)$ is a standard transitive model of T^+ with $W \cap \text{OR} = \infty$, then $\underline{a}^{\widehat{W}} = a$ and $\underline{V}^{\widehat{W}} = V$.

Lemma 3.1. *If $\widehat{W}$ is an outer model of V satisfying T^+ , then satisfaction of atomic $\mathcal{L}^+$ formulas is definable over $\widehat{W}$. Consequently, satisfaction of quantifier-free V -suitable sentences of $\mathcal{L}^+$ is definable over $\widehat{W}$.*

PROOF: Using the predicate symbol $\underline{V}$, the class of (sets representing the) symbols of $\mathcal{L}^+$ is definable over $\widehat{W}$. Note first that the function

$$\text{val}(t(v_1, \dots, v_k), \langle a_1, \dots, a_k \rangle) = t^{\widehat{W}}[a_1, \dots, a_k]$$

that evaluates terms of $\mathcal{L}^+$ in $\widehat{W}$ is definable over $\widehat{W}$ in the usual recursive manner. There are only finitely many constant and function symbols other than the constants $\underline{b}$, for $b \in V$. And their evaluation is uniformly definable from the sets representing them: $\text{val}(\underline{b}, \langle a_1, \dots, a_k \rangle) = b$.

Because there are only finitely many relation symbols, the atomic satisfaction relation is then definable in the usual way. $\square$

The theory T^*

Define an operation Γ_T on sets of V -suitable sentences of $\mathcal{L}^+$:

$$\begin{aligned}
\varphi \in \Gamma_T(X) \quad \text{iff} \quad & \varphi \in X \cup T^+ \text{ or} \\
& \varphi = \neg \exists x \psi, \text{ where } X \cup T^+ \vdash \neg \psi(F(\underline{\alpha})/x), \text{ for all } \alpha < \infty.
\end{aligned}$$

Then Γ_T is a $\Pi_2(V; T^+)$ positive inductive definition. Let T^* be a fixed point of Γ_T . Note that

- $T^+ \subseteq T^*$ and
- if $T^* \vdash \neg \psi(F(\underline{\alpha})/x)$, for all $\alpha < \infty$, then $T^* \vdash \neg \exists x \psi$.

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Proposition 3.2. *If T holds in an outer model W of V such that $(W; \dots, V)$ is admissible, then Γ_T has a consistent fixed point.*

PROOF: Recall that T extends KP + Choice, so Choice holds in W . Forcing over $(W; \dots, V)$ with initial segments of such a function, it is possible to add a KP-preserving, W -amenable class function $G: \text{OR} \rightarrow W$ such that $\text{rng}(G) = W$ and $G(\alpha) \subseteq G''\alpha$, for all $\alpha < \infty$. Expand $(W; \dots, V)$ to a structure $\widehat{W}$ for $\mathcal{L}^+$ by setting $\underline{a}^{\widehat{W}} = a$, for $a \in V$, and $F^{\widehat{W}}(\alpha) = G(\alpha)$, for $\alpha < \infty$. Set $F^{\widehat{W}}(a) = \emptyset$, if $a \in W \setminus \infty$.

Then $\widehat{W} \models T^+$ and $W = \text{rng}(F^{\widehat{W}})$. Let T^* be the set of V -suitable sentences of $\mathcal{L}^+$ that hold in $\widehat{W}$. If $T^* \not\models \neg \exists x \psi$, then $\widehat{W} \models \exists x \psi$. So $\widehat{W} \models \psi(F(\underline{\alpha})/x)$, for some $\alpha < \infty$. Hence $T^* \not\models \neg \psi(F(\underline{\alpha})/x)$. $\square$

The class forcing

For the remainder of this section, let us work under the following

Assumptions.

- T^* is a consistent fixed point of Γ_T .
- T^* is V -amenable and $(V; T^*)$ is admissible.

Working in $(V; T^*)$ define $\mathbb{P}_{T^*}$ to consist of all pairs (r, s) such that

- r is a finite set of basic (that is, atomic and negated atomic) $\mathcal{L}^+$ sentences;
- s is a function such that
 - the domain of s is a natural number;
 - if $n \in \text{dom}(s)$, then $s(n)$ is a finite set of V -suitable sentences;
 - if $m < n < \text{dom}(s)$, then $s(m) \subseteq s(n)$; and
- $T^* \cup r \cup \bigcup \text{rng}(s)$ is consistent.

Order $\mathbb{P}_{T^*}$ by coordinatewise reverse inclusion.

If $(r, s) \in \mathbb{P}_{T^*}$, set

$$\text{th}(r, s) = r \cup \bigcup \text{rng}(s).$$

As before, let $\text{StSpC}(\mathbb{P}_{T^*})$ be the collection of all maximal filters on $\mathbb{P}_{T^*}$. If H is such a filter, set

$$\text{th}(H) = \bigcup_{(r, s) \in H} \text{th}(r, s).$$

If φ is a V -suitable sentence, let

$$D_{\neg \varphi} = \left\{ (r, s) \in \mathbb{P}_{T^*} : \varphi \in \text{th}(r, s) \text{ or } (\neg \varphi) \in \text{th}(r, s) \right\}.$$

Note that $D_{\neg \varphi}$ is dense in $\mathbb{P}_{T^*}$.

If $\bigvee \Phi$ is a quantifier-free V -suitable sentence, set

$$D_{\bigvee \Phi} = \left\{ (r, s) \in \mathbb{P}_{T^*} : \text{either } \neg \bigvee \Phi \in \text{th}(r, s) \text{ or } \varphi \in \text{th}(r, s), \text{ for some } \varphi \in \Phi \right\}.$$

Lemma 3.3. *If $\bigvee \Phi$ is a quantifier-free V -suitable sentence, then $D_{\bigvee \Phi}$ is dense in $\mathbb{P}_{T^*}$.*

PROOF: Let $(\bar{r}, \bar{s})$ be a condition. If $(\bar{r}, \bar{s})$ has no extension (r, s) such that $\neg \bigvee \Phi \in \text{th}(r, s)$, then $T^* \cup \text{th}(\bar{r}, \bar{s}) \models \bigvee \Phi$. If $\mathfrak{A} \models T^* \cup \text{th}(\bar{r}, \bar{s})$, let $\varphi \in \Phi$ be such that $\mathfrak{A} \models \varphi$. Then there exists $(r, s) \leq (\bar{r}, \bar{s})$ such that $\varphi \in \text{th}(r, s)$. $\square$

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If $\exists x\varphi$ is a V -suitable sentence, set

$$D_{\exists x\varphi} = \left\{ (r, s) \in \mathbb{P}_{T^*} : \text{either } (\neg\exists x\varphi) \in \text{th}(r, s) \text{ or } \varphi(F(\underline{\alpha})/x) \in \text{th}(r, s), \text{ for some } \alpha < \infty \right\}.$$

That T^* is a fixed point of Γ_T insures that $D_{\exists x\varphi}$ is dense:

Lemma 3.4. *If $\exists x\varphi$ is a V -suitable sentence, then $D_{\exists x\varphi}$ is dense in $\mathbb{P}_{T^*}$.*

PROOF: Let $(\bar{r}, \bar{s})$ be a condition. We may assume that $\exists x\varphi \in \text{th}(\bar{r}, \bar{s})$. Set $\theta = \bigwedge \text{th}(\bar{r}, \bar{s})$. Then θ is a V -suitable sentence and $T^* \cup \{\exists x(\varphi \wedge \theta)\}$ is consistent. Thus $T^* \not\vdash \neg(\varphi(F(\underline{\alpha})/x) \wedge \theta)$, for some $\alpha < \infty$. Hence $T^* \cup \{\varphi(F(\underline{\alpha})/x), \theta\}$ is consistent, so there exists a condition (r, s) extending $(\bar{r}, \bar{s})$ and such that $\varphi(F(\underline{\alpha})/x) \in \text{th}(r, s)$. $\square$

If H is a filter, say that H is $\neg$ -**complete** if H meets $D_{\neg\varphi}$, for all V -suitable sentences $\neg\varphi$. Define $\bigvee$ -**complete** and $\exists$ -**complete** similarly and say that H is (simply) **complete** if it is $\neg$, $\bigvee$, and $\exists$ -complete.

Theorem 3.5. *Assume that $H \in \text{StSpc}(\mathbb{P}_{T^*})$ is complete. Then there exists a canonical standard $\mathcal{L}^+$ -structure $\widehat{V}_H = (V_H; \dots)$ such that*

- (1) $\widehat{V}_H \models \varphi$ iff $\varphi \in \text{th}(H)$, for all V -suitable sentences φ of $\mathcal{L}^+$;
- (2) $V_H \subseteq V[H]$; and
- (3) $V[H] \subseteq V_H$.

PROOF: **For part (1)**, begin by noting the following six facts:

- (A) $T^* \subseteq \text{th}(H)$
- (B) $\varphi \in \text{th}(H)$ iff $(\neg\varphi) \notin \text{th}(H)$, for all V -suitable sentences φ .
- (C) If $\text{th}(H) \models \varphi$, then $\varphi \in \text{th}(H)$, for all V -suitable sentences φ .
- (D) If $\bigvee \Phi \in \text{th}(H)$, then $\varphi \in \text{th}(H)$, for some $\varphi \in \Phi$.
- (E) If t is a term of $\mathcal{L}^+$ and $\alpha, \beta < \infty$, then
 - $(F(\underline{\alpha}) \doteq F(\underline{\beta})) \in \text{th}(H)$;
 - if $(F(\underline{\alpha}) \doteq F(\underline{\beta})) \in \text{th}(H)$, then $(F(\underline{\beta}) \doteq F(\underline{\alpha})) \in \text{th}(H)$; and
 - if $\varphi(t), (F(\underline{\alpha}) \doteq t) \in \text{th}(H)$, then $\varphi(F(\underline{\alpha})/t) \in \text{th}(H)$.
- (F) If $\forall x\varphi \in \text{th}(H)$, then $\varphi(F(\underline{\alpha})/x) \in \text{th}(H)$, for all $\alpha < \infty$.

Proposition (D) uses that H is $\bigvee$ -complete. The others just use that H is $\neg$ -complete and that $T^* \cup \text{th}(r, s)$ is consistent, for $(r, s) \in H$.

Declare that

$$F(\underline{\alpha}) \equiv F(\underline{\beta}) \quad \text{iff} \quad (F(\underline{\alpha}) \doteq F(\underline{\beta})) \in \text{th}(H).$$

Then $\equiv$ is an equivalence relation on the closed terms $\{F(\underline{\alpha}) : \alpha < \infty\}$ by (E). Let $[F(\underline{\alpha})]$ denote the equivalence class of $F(\underline{\alpha})$ and let A be the collection of these equivalence classes. Using (E) to insure to see that this is well-defined, interpret the relations and constants of $\mathcal{L}^+$ in the natural manner to obtain a structure $\mathfrak{A}$ for $\mathcal{L}^+$ with the property that

- (1) $\mathfrak{A} \models \varphi$ iff $\varphi \in \text{th}(H)$,

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for all atomic sentences φ . Then a straightforward induction on quantifier-free sentences using (B)–(D) establishes that line (1) holds for all quantifier-free V -suitable sentence of $\mathcal{L}^+$.

In order to see that $\hat{\in}^{\mathfrak{A}}$ is well-founded, it suffices to see that if $[F(\underline{\gamma})] \hat{\in}^{\mathfrak{A}} [F(\underline{\alpha})]$, then there exists $\beta < \alpha$ such that $F(\underline{\gamma}) \equiv F(\underline{\beta})$. Note that

$$\forall x (x \hat{\in} F(\underline{\alpha}) \rightarrow \bigvee_{\beta < \alpha} x \hat{=} F(\underline{\beta})) \in T^+ \subseteq \text{th}(H),$$

so $\mathfrak{A} \models \bigvee_{\beta < \alpha} F(\underline{\gamma}) \hat{=} F(\underline{\beta})$ by (F). Hence $(F(\underline{\gamma}) \hat{=} F(\underline{\beta})) \in \text{th}(H)$, for some $\beta < \alpha$.

Let $\hat{V}_H = (V_H; \cdots)$ be the standard transitive structure isomorphic to $\mathfrak{A}$.

To finish the proof of part (1), note that if H is also $\exists$ -complete, then we have as well that

(G) if $\exists x \varphi \in \text{th}(H)$, then $\varphi(F(\underline{\alpha})/x) \in \text{th}(H)$, for some $\alpha < \infty$.

Using this, the induction on quantifier-free sentences that established line (1) can be extended to all V -suitable sentences of $\mathcal{L}^+$.

A consequence of (1) is that V_H end-extends V , since $\hat{V}_H$ is a standard transitive model of T^+ . Indeed, $\underline{a}^{\hat{V}_H} = a$, for $a \in V$.

For part (2), working in $(V; T^*)$, define a function $F^*: \text{OR} \rightarrow V^{\mathbb{P}_{T^*}}$ by recursion on $\alpha < \infty$:

$$(2) \quad F^*(\alpha) = \left\{ \left(F^*(\beta), (\{F(\underline{\beta}) \hat{\in} F(\underline{\alpha})\}, \emptyset) \right) : \right. \\ \left. \beta < \alpha \text{ and } T^* \cup \{F(\underline{\beta}) \hat{\in} F(\underline{\alpha})\} \text{ is consistent} \right\}$$

Note first that for any $\beta, \alpha < \infty$,

$$\begin{aligned} F(\underline{\beta})^{\hat{V}_H} \in F(\underline{\alpha})^{\hat{V}_H} & \text{ iff } (F(\underline{\beta}) \hat{\in} F(\underline{\alpha})) \in \text{th}(H) \\ & \text{ iff } \exists (r, s) \in H \ (F(\underline{\beta}) \hat{\in} F(\underline{\alpha})) \in \text{th}(r, s) \\ & \text{ iff } \exists (r, s) \in H \ (F(\underline{\beta}) \hat{\in} F(\underline{\alpha})) \in r, \text{ since } H \text{ is maximal} \\ & \text{ iff } (\{F(\underline{\beta}) \hat{\in} F(\underline{\alpha})\}, \emptyset) \in H. \end{aligned}$$

Now proceed by induction on α to see that $F^*(\alpha)^H = F(\underline{\alpha})^{\hat{V}_H}$. First note that if $F(\underline{\gamma})^{\hat{V}_H} \in F(\underline{\alpha})^{\hat{V}_H}$, then $F(\underline{\gamma})^{\hat{V}_H} = F(\underline{\beta})^{\hat{V}_H}$, for some $\beta < \alpha$. Using the above equivalence, it follows that $F^*(\beta)^H \in F^*(\alpha)^H$. And $F^*(\beta)^H = F(\underline{\beta})^{\hat{V}_H}$ by induction. Hence $F(\underline{\alpha})^{\hat{V}_H} \subseteq F^*(\alpha)^H$.

For the converse inclusion, suppose that $F^*(\gamma)^H \in F^*(\alpha)^H$. Say $F^*(\gamma)^H = F^*(\beta)^H$, where $\beta < \alpha$ and $(\{F(\underline{\beta}) \hat{\in} F(\underline{\alpha})\}, \emptyset) \in H$. Then $F(\underline{\beta})^{\hat{V}_H} \in F(\underline{\alpha})^{\hat{V}_H}$ by the above equivalence and $F^*(\beta)^H = F(\underline{\beta})^{\hat{V}_H}$ by induction.

A consequence of (1) and (2) is that V_H is an outer model of V . Indeed, $\underline{V}^{\hat{V}_H} = V$.

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For part (3)—and this notion will figure in later developments, as well—define a Shoenfield term $\mathring{a}$ to be *special* if, hereditarily, it only involves conditions of the form $(r, \emptyset)$. That is,

$$\mathring{a} \text{ is special} \quad \text{iff} \quad \mathring{c} \text{ is special and } s = \emptyset, \text{ whenever } (\mathring{c}, (r, s)) \in \mathring{a}.$$

The point of special terms is that (1) in any modestly generic extension, every element of the universe is denoted by a special term, and (2) special terms can be evaluated using just the basic diagram of the extension. More precisely, this is the content of the following two claims.

Claim 1. *Assuming just that H is $\neg$ -complete, given any Shoenfield term $\mathring{a}$, there exists a special term $\mathring{a}^*$ such that $\mathring{a}^H = \mathring{a}^{*H}$.*

PROOF: By $\neg$ -completeness, there exists a condition $(r, s) \in H$ such that $\bigcup \text{rng}(s) \setminus \text{TC}(\{\mathring{a}\}) \neq \emptyset$. Note that if $(r', s') \in \text{TC}(\{\mathring{a}\})$ is compatible with (r, s) , then s' is an initial segment of s . By recursion on Shoenfield terms $\mathring{b} \in \text{TC}(\{\mathring{a}\})$ define

$$\mathring{b}^* = \left\{ (\mathring{c}^*, (r', \emptyset)) : (\mathring{c}, (r', s')) \in \mathring{b} \text{ and } (r', s') \text{ is compatible with } (r, s) \right\}.$$

Then $\mathring{b}^*$ is a special term. Furthermore,

$$(r', \emptyset) \in \mathring{b}^*(\mathring{c}^*) \cap H \quad \text{iff} \quad (r', s') \in \mathring{b}(\mathring{c}) \cap H, \text{ for some } s'.$$

It follows by induction on Shoenfield terms $\mathring{b}$ in $\text{TC}(\{\mathring{a}\})$ that $\mathring{b}^H = \mathring{b}^{*H}$. $\square$

Claim 2. *If $\mathring{a}$ is a special term, then $\mathring{a}^H \in V_H$.*

PROOF: First note that

$$\left\{ \mathring{b} \in \text{TC}(\{\mathring{a}\}) : \mathring{b} \text{ is a special term} \right\} \in V \subseteq V_H$$

and that

$$(r, \emptyset) \in H \quad \text{iff} \quad \widehat{V}_H \models r$$

by part (1). By part (1) and Lemma 3.1 we have that satisfaction of atomic sentences of $\mathcal{L}^+$ is definable over $\widehat{V}_H$. Hence, $\{(r, \emptyset) : (r, \emptyset) \in H\}$ is definable over $\widehat{V}_H$. It follows that $\mathring{b}^H$ is definable by recursion on special terms $\mathring{b} \in \text{TC}(\{\mathring{a}\})$, working in $\widehat{V}_H$. $\square$

4. Complete filters and genericity

Let us continue under the Assumption of the previous section, namely, that

- T^* is a V -amenable fixed point of Γ_T , and that
- $(V; T^*)$ is admissible.

In this section we shall see that any complete filter on $\mathbb{P}_{T^*}$ is generic. To do this, first we define a translation procedure $\varphi \mapsto \varphi^*$ from sentences φ in a rich enough fragment of the forcing language to V -suitable sentences φ^* . The connection between logical completeness and genericity is made by showing that $(r, s) \Vdash \varphi$ iff $T^* \cup \text{th}(r, s) \vdash \varphi^*$.

Say that a model $\mathfrak{A}$ of T^* is *eunonymous* if every element of the universe of $\mathfrak{A}$ lies in the range of $F^{\mathfrak{A}}$.

Proposition 4.1. *Suppose that $\mathfrak{A} = (A; \dots)$ is a eunonymous model of T^* . Then there exists a complete filter $H \in \text{StSpc}(\mathbb{P}_{T^*})$ such that $(V_H; \dots) \cong \mathfrak{A}$.*

PROOF: Let f be a function with domain ω such that

- $f(n)$ is a finite set of V -suitable sentences that hold in $\mathfrak{A}$;
- if $m \leq n$, then $f(m) \subseteq f(n)$; and
- every V -suitable sentence that holds in $\mathfrak{A}$ lies in $\bigcup \text{rng}(f)$.

Set

$$H = \left\{ (r, s) \in \mathbb{P}_{T^*} : \mathfrak{A} \models r \text{ and } s = f \upharpoonright n, \text{ for some } n \in \omega \right\}.$$

Then H is a complete filter in $\text{StSpc}(\mathbb{P}_{T^*})$. Furthermore, for V -suitable sentences φ of $\mathcal{L}^+$,

$$(V_H; \dots) \models \varphi \quad \text{iff} \quad \varphi \in \text{th}(H) \quad \text{iff} \quad \mathfrak{A} \models \varphi.$$

Then $(V_H; \dots) \cong \mathfrak{A}$, since both $(V_H; \dots)$ and $\mathfrak{A}$ are eunonymous. $\square$

Because T^* may not be definable in $(V_H; \dots)$, we need to be careful to avoid reference to it in the following definitions. The class of *pseudo-conditions* is defined exactly like $\mathbb{P}_{T^*}$, except that the requirement that $T^* \cup \text{th}(r, s)$ be consistent is weakened to require only that $T^+ \cup \text{th}(r, s)$ be consistent. So every condition is a pseudo-condition, but not conversely. The class of pseudo-conditions is definable over V . Bad pseudo-conditions will turn out to be harmless to our ends because $(V_H; \dots)$ satisfies T^* .

Literally, $\mathbb{P}_{T^+}$ is the class of pseudo-conditions, but it is best not to rely on such delicate distinctions in notation.

A Shoenfield *pseudo-term* is a Shoenfield term for the class of pseudo-conditions. Every Shoenfield term of $V^{\mathbb{P}_{T^*}}$ is a pseudo-term, but not conversely. The *pseudo-forcing language* is the same as the forcing language for $\mathbb{P}_{T^*}$ except that it has pseudo-terms as constant symbols as well as true Shoenfield terms. The value $\dot{a}^H$ of pseudo-terms is defined exactly as is the value of terms.

A *special sentence* of the pseudo-forcing language is one in which only special pseudo-terms occur. A *special pseudo-term* is a pseudo-term that, hereditarily, only involves pseudo-conditions of the form $(r, \emptyset)$.

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Let us define in V a translation procedure $\varphi \mapsto \varphi^*$ from prenex special sentences φ of the pseudo-forcing language to V -suitable sentences φ^* of the language $\mathcal{L}^+$. We shall first define this procedure for quantifier-free φ , then for general prenex φ .

First, proceed by induction on quantifier-free special sentences of the pseudo-forcing language:

$$\begin{aligned} (\dot{a} \hat{=} \dot{b})^* &= \bigvee_{\dot{c} \in \text{dom}(\dot{b})} \left(\bigvee_{(r, \emptyset) \in \dot{b}(\dot{c})} \bigwedge r \wedge (\dot{c} \hat{=} \dot{a})^* \right) \\ (\dot{b} \hat{=} \dot{a})^* &= \bigwedge_{\dot{c} \in \text{dom}(\dot{b})} \left(\bigvee_{(r, \emptyset) \in \dot{b}(\dot{c})} \bigwedge r \rightarrow (\dot{c} \hat{=} \dot{a})^* \right) \\ (\varphi \wedge \psi)^* &= (\varphi^* \wedge \psi^*) \\ (\neg \varphi)^* &= (\neg \varphi^*) \end{aligned}$$

Our convention is that an empty conjunction is a (fixed) tautology in the language of set theory and that an empty disjunction is a (fixed) contradiction in the language of set theory.

Note that this definition yields a definable operation in V ,

$$\left(\varphi(x_1, \dots, x_k), \langle \dot{a}_1, \dots, \dot{a}_k \rangle \right) \mapsto \varphi(\dot{a}_1/x_1, \dots, \dot{a}_k/x_k)^*$$

for quantifier-free (finitary) formulas $\varphi(x_1, \dots, x_k)$ in the language $\{\hat{=}, \hat{=}\}$ and sequences $\langle \dot{a}_1, \dots, \dot{a}_k \rangle$ of special pseudo-terms. Note that the output of this operation consists of quantifier-free infinitary sentences in the language $\mathcal{L}^+$.

Recall the function F^* defined in line (2) of section 3:

$$F^*(\alpha) = \left\{ \left(F^*(\beta), (\{F(\underline{\beta}) \hat{=} F(\underline{\alpha})\}, \emptyset) \right) : \right. \\ \left. \beta < \alpha \text{ and } T^* \cup \{F(\underline{\beta}) \hat{=} F(\underline{\alpha})\} \text{ is consistent.} \right\}$$

It was shown there that $F^*(\alpha)$ is a Shoenfield term with the property that if $H \in \text{StSpc}(\mathbb{P}_{T^*})$, then $F^*(\alpha)^H = F(\underline{\alpha})^{(V_H; \dots)}$.

In analogy, define a function F^{**} from ordinals into pseudo-terms:

$$F^{**}(\alpha) = \left\{ \left(F^{**}(\beta), (\{F(\underline{\beta}) \hat{=} F(\underline{\alpha})\}, \emptyset) \right) : \beta < \alpha \right\}.$$

Note that F^{**} is definable over V .

Lemma 4.2. *If $H \in \text{StSpc}(\mathbb{P}_{T^*})$, then $F^*(\alpha)^H = F^{**}(\alpha)^H$.*

PROOF: Proceed by induction on α . The inclusion $F^*(\alpha)^H \subseteq F^{**}(\alpha)^H$ is immediate. For the converse inclusion, suppose that $b \in F^{**}(\alpha)^H$. Then $b = F^{**}(\beta)^H$, for some $\beta < \alpha$ such that $(F(\underline{\beta}) \hat{=} F(\underline{\alpha})) \in \text{th}(H)$. But $\text{th}(H) \supseteq T^*$ is consistent, so $(F^*(\beta), (\{F(\underline{\beta}) \hat{=} F(\underline{\alpha})\}, \emptyset)) \in F^*(\alpha)$. Hence $F^*(\beta)^H \in F^*(\alpha)^H$. And $b = F^{**}(\beta)^H = F^*(\beta)^H$ by induction. $\square$

4. COMPLETE FILTERS AND GENERICITY

Let $\text{Sat}(z)$ be a formula uniformly defining satisfaction of quantifier-free V -suitable sentences of $\mathcal{L}^+$ over outer models of V satisfying T^+ . Using the V -definable $*$ -operation and Sat , extend the $*$ -operation to special prenex sentences of the forcing language:

$$\begin{aligned} \left(Q_1 x_1 \cdots Q_k x_k \psi \right)^* = \\ Q_1 \alpha_1 \hat{\in} \text{OR} \cdots Q_k \alpha_k \hat{\in} \text{OR} \text{ Sat} \left(\psi(F^{**}(\alpha_1)/x_1, \dots, F^{**}(\alpha_k)/x_k)^* \right). \end{aligned}$$

Proposition 4.3. *Assume that H is a complete filter in $\text{StSpc}(\mathbb{P}_{T^*})$. Then*

$$V[H] \models \varphi \quad \text{iff} \quad (V_H; \cdots) \models \varphi^*,$$

for all special prenex sentences φ of the forcing language.

PROOF: First, proceed by induction to prove this equivalence for quantifier-free special sentences:

$$\begin{aligned} V[H] \models \dot{a} \hat{\in} \dot{b} \quad & \text{iff} \quad \dot{a}^H \in \dot{b}^H \\ & \text{iff} \quad \exists \dot{c} \in \text{dom}(\dot{b}) \exists (r, \emptyset) \in \dot{b}(\dot{c}) \left((r, \emptyset) \in H \text{ and } \dot{a}^H = \dot{c}^H \right) \\ & \text{iff} \quad \exists \dot{c} \in \text{dom}(\dot{b}) \exists (r, \emptyset) \in \dot{b}(\dot{c}) \left((V_H; \cdots) \models r \text{ and } (V_H; \cdots) \models (\dot{a} \hat{=} \dot{c})^* \right) \\ & \text{iff} \quad (V_H; \cdots) \models \bigvee_{\dot{c} \in \text{dom}(\dot{b})} \left(\bigvee_{(r, \emptyset) \in \dot{b}(\dot{c})} r \wedge (\dot{a} \hat{=} \dot{c})^* \right) \\ & \text{iff} \quad (V_H; \cdots) \models (\dot{a} \hat{\in} \dot{b})^* \end{aligned}$$

The other cases are similar.

Now if $\psi(x_1, \dots, x_k)$ is a quantifier-free special formula of the forcing language, then the following statements are equivalent:

- (1) $V[H] \models Q_1 x_1 \cdots Q_k x_k \psi$
- (2) $Q_1 \alpha_1 < \infty \cdots Q_k \alpha_k < \infty \ V[H] \models \psi(F^*(\alpha_1)/x_1, \dots, F^*(\alpha_k)/x_k)$
- (3) $Q_1 \alpha_1 < \infty \cdots Q_k \alpha_k < \infty \ V[H] \models \psi(F^{**}(\alpha_1)/x_1, \dots, F^{**}(\alpha_k)/x_k)$
- (4) $Q_1 \alpha_1 < \infty \cdots Q_k \alpha_k < \infty \ (V_H; \cdots) \models \psi(F^{**}(\alpha_1)/x_1, \dots, F^{**}(\alpha_k)/x_k)^*$
- (5) $(V_H; \cdots) \models Q_1 \alpha_1 \hat{\in} \text{OR} \cdots Q_k \alpha_k \hat{\in} \text{OR} \text{ Sat} \left(\psi(F^{**}(\alpha_1)/x_1, \dots, F^{**}(\alpha_k)/x_k)^* \right)$

The equivalence of (1) and (2) uses that V_H is eunonymous, that $V[H] = V_H$, and that $F^*(\alpha)^H = F(\underline{\alpha})^{(V_H; \cdots)}$. The equivalence of (2) and (3) uses Lemma 4.2. The equivalence of (4) and (5) uses that $\text{OR}^{V_H} = \infty$. $\square$

Corollary 4.4. *If φ is a special prenex sentence of the forcing language, then*

$$(r, s) \Vdash \varphi \quad \text{iff} \quad T^* \cup \text{th}(r, s) \vdash \varphi^*.$$

PROOF: ($\Leftarrow$) Assume that $T^* \cup \text{th}(r, s) \vdash \varphi^*$. Because $(r, s) \Vdash \varphi$ iff $(r, s) \models \varphi$, and because complete H are comeager in $\text{StSpc}(\mathbb{P}_{T^*})$, it suffices to see that if H is complete and $(r, s) \in H$, then $V[H] \models \varphi$. By the previous proposition, it suffices to see that $(V_H; \cdots) \models \varphi^*$. But $T^* \cup \text{th}(r, s) \subseteq \text{th}(H)$, so $\text{th}(H) \vdash \varphi^*$; hence $\varphi^* \in H$; hence $(V_H; \cdots) \models \varphi^*$.

($\Rightarrow$) Assume that $T^* \cup \text{th}(r, s) \not\vdash \varphi^*$. Then there exists a condition (r', s') below (r, s) such that $\neg \varphi^* \in \text{th}(r', s')$. Then $(r', s') \Vdash \neg \varphi$ by the previous paragraph. Consequently $(r, s) \not\Vdash \varphi$. $\square$

Theorem 4.4. *Assume that H is a complete filter in $\text{StSpc}(\mathbb{P}_{T^*})$. Then H is generic over $(V; \mathbb{P}_{T^*})$.*

PROOF: Let $\psi(\dot{a}_1, \dots, \dot{a}_k)$ be a sentence in the forcing language. Let $\dot{b}_1, \dots, \dot{b}_k$ be special terms (as in Claim 1 in the proof of Theorem 3.5) such that $\dot{a}_i^H = \dot{b}_i^H$, for all i , and let $\varphi(\dot{b}_1, \dots, \dot{b}_k)$ be a prenex sentence equivalent to $\psi(\dot{b}_1, \dots, \dot{b}_k)$.

Assume that $V[H] \models \psi(\dot{a}_1, \dots, \dot{a}_k)$. Then $V[H] \models \varphi(\dot{b}_1, \dots, \dot{b}_k)$. Hence $(V_H; \dots) \models \varphi(\dot{b}_1, \dots, \dot{b}_k)^*$ and so $\varphi(\dot{b}_1, \dots, \dot{b}_k)^* \in \text{th}(H)$. Thus there is a condition $(r, s) \in H$ with $\varphi(\dot{b}_1, \dots, \dot{b}_k)^* \in \text{th}(r, s)$. Then $(r, s) \Vdash \varphi(\dot{b}_1, \dots, \dot{b}_k)$. By the proof of Claim 1 of Theorem 3.5, we may assume that $(r, s) \Vdash \dot{b}_i \dot{=} \dot{a}_i$, for all i . It follows that $(r, s) \Vdash \psi(\dot{a}_1, \dots, \dot{a}_k)$.

Similarly, if $V[H] \not\models \psi(\dot{a}_1, \dots, \dot{a}_k)$, then some condition (r, s) in H forces that $\neg\psi(\dot{a}_1, \dots, \dot{a}_k)$, hence no condition in H forces $\psi(\dot{a}_1, \dots, \dot{a}_k)$. $\square$

5. The main theorems

Let us begin with an easy consequence of our work to this point.

Theorem 5.1. *Assume that*

- V is a countable standard model of ZFC;
- $T \in V$ is a set of axioms in a finite language extending $KP + \text{Choice}$;
- T^* is a V -amenable fixed point of Γ_T such that
- $(V; T^*)$ is admissible;
- W is an outer model of V ; and
- W has an expansion $\widehat{W}$ that is a eunonymous model of T^* .

Then there exists a filter $G \in \text{StSpc}(\mathbb{P}_{T^})$ such that*

- G is generic over $(V; \mathbb{P}_{T^*})$ and
- $W = V[G]$.

PROOF: There exists a complete filter G on $\mathbb{P}_{T^*}$ such that $\widehat{W} \cong (V_G; \dots)$ by Proposition 4.1. But W and V_G are transitive, so they are equal. In turn, by Theorem 3.5 we have that $V_G = V[G]$ and, by Theorem 4.4, that G is generic. $\square$

Using Theorem 5.1, we can now prove Theorem 1.2 of section 1.

First, recall the definition of “sufficiently non-minimal” given there: Assume that there exists a global well-ordering of the universe V , so that we can calculate Skolem hulls uniformly. If, for example, $V = L$, then such a well-ordering is definable. In any case, an amenable well-ordering of the universe can be added by a simple class forcing and we can work relative to a predicate for this well-ordering.

If κ is a cardinal, let $\text{Hyp}(H_\kappa)$ be the smallest admissible set with H_κ as an element.

$$u(\kappa) = \left\{ \lambda \leq \kappa : \lambda \text{ is a cardinal and } H_\lambda = \text{Skolem Hull}_{\text{Hyp}(H_\kappa)}(H_\lambda) \cap H_\kappa \right\}.$$

Then $u(\kappa) \subseteq \kappa + 1$ is a closed set of cardinals with $\kappa \in u(\kappa)$. Set

$$U = \left\{ u(\kappa) : \kappa \text{ is a cardinal} \right\}.$$

Then U is a definable tree under end-extension.

5. THE MAIN THEOREMS

Theorem 1.2. *Suppose that W is a countable standard transitive outer model of V , and that there exists a branch B through U such that $\sup(B) = \infty = W \cap \text{OR}$ and $(W; V, B)$ satisfies ZFC. Then there exists a $(V; B)$ -definable partial ordering $\mathbb{P}$ and a filter H on $\mathbb{P}$ such that H is generic over $(V; \mathbb{P})$ and $W = V[H]$.*

PROOF: Work in $(V; B)$. Let $T = \text{KP} + \text{Choice}$. Note that if $\kappa \in B$, then $H_\kappa = V_\kappa$. We may assume that symbols and formulas of $\mathcal{L}^+$ are represented by sets in a straightforward manner so that if $\kappa \in B$, then $(T^+)^{H_\kappa}$ is the following set of axioms:

$$\begin{aligned} & \text{KP} + \text{Choice in the language of set theory} \\ & \text{KP in the language } \{\hat{=}, F, \underline{V}\} \cup \{\underline{a} : a \in H_\kappa\} \\ & \forall x (x \hat{=} \underline{a} \leftrightarrow \bigvee_{b \in a} x \hat{=} \underline{b}), \text{ for } a \in H_\kappa \\ & \underline{V} \underline{a}, \text{ for } a \in H_\kappa \\ & \forall x (\underline{V}x \wedge \text{rk}(x) < \underline{\alpha} \rightarrow \bigvee_{a \in V_\alpha} x \hat{=} \underline{a}), \text{ for } \alpha < \kappa \\ & \forall x (x \not\hat{=} \text{OR} \rightarrow F(x) \hat{=} \emptyset) \\ & \forall x (x \hat{=} F(\underline{\alpha}) \rightarrow \bigvee_{\beta < \alpha} x \hat{=} F(\underline{\beta})), \text{ for } \alpha < \kappa. \end{aligned}$$

Let T^κ be the least fixed point of $(\Gamma_T)^{H_\kappa}$. Then $T^\kappa \subseteq H_\kappa$ is Σ_1 -definable over $\text{Hyp}(H_\kappa)$. Hence if $\lambda < \kappa$ lie in B , then $T^\lambda = T^\kappa \cap H_\lambda$. Set $T^* = \bigcup_{\kappa \in B} T^\kappa$. Using Barwise Completeness, we have that T^* is a fixed point of Γ_T , though perhaps not the least fixed point, since the direct limit of $\langle \text{Hyp}(H_\kappa) : \kappa \in B \rangle$ may be ill-founded. Furthermore, T^* is (V, B) -definable and $(V; B)$ satisfies ZFC. By the previous theorem, it suffices to see that W has a eunonymous expansion satisfying T^* .

Let $G: \text{OR} \rightarrow W$ be a W -amenable enumeration of W such that $(W; V, B, G)$ satisfies ZFC and $G(\alpha) \subseteq G''\alpha$, for all $\alpha < \infty$. Since $(W; V, B)$ satisfies ZFC, it is possible to find such a G by $(W; V, B)$ -definable class forcing.

Extend G to $G': W \rightarrow W$ by setting $G'(a) = \emptyset$, for $a \in W \setminus \infty$. We maintain that $\widehat{W} = (W; V, G', a)_{a \in V}$ satisfies T^* . Note that there exists a $(\widehat{W}; B)$ -definable club class of κ such that $\kappa \in B$ and $H_\kappa^W = G''\kappa$ and $\widehat{W}_\kappa \models \text{KP}$, where $\widehat{W}_\kappa = (H_\kappa^W; H_\kappa^V, G' \upharpoonright \kappa, a)_{a \in H_\kappa^V}$. It follows that $\widehat{W}_\kappa \models (T^+)^{H_\kappa^V}$. Using that $H_\kappa^W = G''\kappa$, the set of H_κ^V -suitable sentences holding in $\widehat{W}_\kappa$ is a fixed point of $(\Gamma_T)^{H_\kappa^V}$. Hence every sentence in T^κ , the *least* fixed point of $(\Gamma_T)^{H_\kappa^V}$, holds in $\widehat{W}_\kappa$. It follows that every sentence in T^* holds in $\widehat{W}$. $\square$

If $V = L$ and W is an outer model of $\text{ZFC} + "0^\# \text{ exists}"$, then

$$B = \left\{ u(\kappa) : \kappa \text{ is a Silver indiscernible} \right\}$$

is a branch through U with $\sup(B) = \infty$. Consequently, we have the following

Corollary 5.3. *Suppose that W is a countable standard model of $\text{ZFC} + "0^\# \text{ exists}"$. Then there exists a W -definable, L -amenable partial ordering $\mathbb{P}$ and a filter H such that H is generic over $(L; \mathbb{P})$ and $V = L[H]$.*

6. Preserving ZFC in a larger language

If G is generic over $(V; \mathbb{P})$ for the forcing $\mathbb{P}$ of Theorem 1.2, then $(V[G]; G)$ does not satisfy ZFC. This cannot be fixed without some additional hypotheses or a substantially different proof. A key feature of the proof is that truth in $V[G]$ is easily defined from G . Consequently, if $W = V[G]$ is a minimal model, then $(V[G]; G)$ cannot satisfy ZFC.

For Theorem 1.2 we needed the inner model V to be sufficiently non-minimal; indeed, we needed a witness to this non-minimality compatible with ZFC in the outer model. Hugh Woodin observed that if we assume the *outer* model W to be sufficiently non-minimal, then it is possible to get an improved version of Theorem 1.2 in which $(V[G]; V, \mathbb{P}, G) \models \text{ZFC}$. In general, it is still only with respect to the language of set theory that G is generic, not with respect to the enlarged language with symbols for these auxiliary predicates.

Woodin's idea is to use an infinitary language in which sentences are ramified in a hierarchy of length $\infty = \text{OR} \cap W$, rather than the Σ_n hierarchy of V -suitable formulas used in Theorem 1.2.

Woodin's result can be established by modifying the proof of Theorem 1.2. Let us work under the assumption that ∞ is a definably regular cardinal in $\text{Hyp}(W; V)$, that is, that there is no $\text{Hyp}(W; V)$ -definable function from an ordinal less than ∞ with range cofinal in ∞ . In fact, a bit weaker non-minimality hypothesis supports the same proof, but there is no need to pursue that here.

As before, a generic filter G builds a Henkin model $\widehat{V}_G$, working in a certain infinitary language. Again, under mild hypotheses, $V_G \subseteq V[G]$. The converse inclusion uses that the value $\hat{a}^G$ of the Shoenfield term $\hat{a}$ only depends on infinitary sentences of bounded complexity. This is because $\hat{a} \in V$ and the complexity hierarchy has length ∞ .

Again, if G is complete in the sense that it decides all sentences of the infinitary language, then G is generic over $(V; \mathbb{P})$ with respect to the language of set theory. The proof of this last claim again uses that approximations to the satisfaction predicate for $\widehat{V}_G$ are definable over $\widehat{V}_G$.

The reason that $(V[G]; V, \mathbb{P}, G)$ satisfies ZFC is that all of these extra predicates are definable over $\text{Hyp}(W; V)$ and ∞ is definably regular there.

Theorem 1.5. (Woodin) *Assume that W is a countable outer model of V and that $\infty = \text{OR} \cap W$ is definably regular in $\text{Hyp}(W; V)$. There exists a partial ordering $\mathbb{P}$ and a filter G such that*

- $\mathbb{P}$ is V -amenable and definable over $\text{Hyp}(V)$;
- G is generic over $(V; \mathbb{P})$ with respect to the language of set theory;
- $W = V[G]$; and
- $(V[G]; V, \mathbb{P}, G) \models \text{ZFC}$.

PROOF: Let

$$\mathcal{L}^+ = \{\hat{\in}, \underline{V}, F\} \cup \{\underline{a} : a \in V\} \cup \{S_\alpha : \alpha < \infty\},$$

where, as before,

- $\hat{\in}$ is a binary relation symbol (for the membership relation);

6. PRESERVING ZFC IN A LARGER LANGUAGE

- $\underline{V}$ is a unary predicate symbol (for V as an inner model);
- F is a unary function symbol (the terms $F(\underline{\alpha})$ serve as Henkin constants);
- $\underline{a}$ is a constant symbol (for $a \in V$);

and, in addition,

- S_α is a unary predicate symbol (for an unbounded class of ordinals δ such that $F(\delta) = W_\delta$ and W_δ is an elementary substructure of W relative to S_β , for $\beta < \alpha$, as well as $\hat{e}$, $\underline{V}$, and $\underline{a}$, for $a \in V_\delta$).

Work in the infinitary fragment $\mathcal{L}_V^+$, that is, those formulas of $\mathcal{L}_{\infty\omega}^+$ that (are coded by sets that) lie in V . We assume that the formulas of $\mathcal{L}_V^+$ are coded by sets in such a way that

$$\mathcal{L}^+ \cap V_\delta = \left\{ \hat{e}, \underline{V}, F \right\} \cup \left\{ \underline{a} : a \in V_\delta \right\} \cup \left\{ S_\alpha : \alpha < \delta \right\},$$

when δ is a limit ordinal. Let $\mathcal{L}_{V_\delta}^+$ be $\mathcal{L}_V^+ \cap V_\delta$.

As before, T^+ is a set of axioms enforcing the intended interpretations of the symbols in eunonymous outer models $\hat{U}$. Recall that an outer model $\hat{U}$ is *eunonymous* if $F^{\hat{U}}$ maps ∞ onto U . The first five groups of axioms in T^+ are essentially as before. The last three, which relate to the S_α 's, are new. Let T^+ be formalizations in $\mathcal{L}_V^+$ of the following statements:

- ZFC in $\mathcal{L}^+$
- $\forall x (x \hat{e} \underline{a} \leftrightarrow \bigvee_{b \in a} x \hat{e} \underline{b})$, for each $a \in V$
- $\forall x (\underline{V}x \wedge \text{rk}(x) < \underline{\alpha} \leftrightarrow x \hat{e} \underline{V}_\alpha)$, for each $\alpha < \infty$
- $\forall x (x \notin \text{OR} \rightarrow F(x) = \emptyset)$
- $\forall x (x \hat{e} F(\underline{\alpha}) \rightarrow \bigvee_{\beta < \alpha} x \hat{e} F(\underline{\beta}))$, for each $\alpha < \infty$
- “ S_α is a set of limit ordinals and is unbounded in ∞ ”, for each $\alpha < \infty$
- $\forall \delta (S_\alpha \delta \rightarrow \forall x (x \hat{e} F(\delta) \leftrightarrow \text{rk}(x) < \delta))$
[So $F^{\hat{U}}(\delta) = U_\delta$, when $\hat{U}$ satisfies T^+ .]
- $\forall \delta (S_\alpha \delta \rightarrow \bigwedge_{\varphi \text{ a sentence of } \mathcal{L}_{V_\delta}^+} (\varphi^{F(\delta)} \leftrightarrow \varphi))$

[If $\hat{U}$ is eunonymous and satisfies T^+ and δ lies in $S_\alpha^{\hat{U}}$, then $\hat{U}_\delta$ is an elementary substructure of $\hat{U}$ in the language $\mathcal{L}_{V_\delta}^+$, which involves S_β , for $\beta < \alpha$, as well as $\hat{e}$, F , and $\underline{a}$, for $a \in V_\delta$.]

The outer model W has a eunonymous expansion $\hat{W}$ satisfying T^+ ; indeed, without loss of generality, we may assume that $\hat{W}$ is an element of $\text{Hyp}(W; V)$. To see this, note that the required interpretations of $\hat{e}$, $\underline{V}$, and $\underline{a}$, for $a \in V$, are clearly in $\text{Hyp}(W; V)$. The hypothesis that ∞ is definably regular in $\text{Hyp}(W; V)$ insures that suitable interpretations of the S_α 's—interpretations preserving ZFC—lie in $\text{Hyp}(W; V)$, provided a suitable interpretation for F does. Because there may be no global well-ordering of W in $\text{Hyp}(W; V)$, set forcing (with a $< \infty$ closed tree) over $\text{Hyp}(W; V)$ may be required to add $F^{\hat{W}}$. Such a generic $F^{\hat{W}}$ does exist, because W is countable. Since this set forcing preserves all of the relevant properties of $\text{Hyp}(W; V)$, let us simply assume that $F^{\hat{W}} \in \text{Hyp}(W; V)$.

In analogy to the Lévy hierarchy, say that an $\mathcal{L}_V^+$ sentence is a Σ_α sentence if the predicate symbols S_β occurring in it are among $\{S_\beta : \beta < \alpha\}$. Note that in a

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eunonymous outer model of V satisfying T^+ , satisfaction of Σ_α sentences φ is definable by the formula

$$\exists \delta \left(S_\alpha \delta \wedge \ulcorner \varphi \urcorner \in V_\delta \wedge \langle F(\delta); F \upharpoonright F(\delta), V_\delta, a, S_\beta \cap \delta \rangle_{a \in V_\delta, \beta < \alpha} \text{ satisfies } \varphi \right).$$

For short, let “ $\text{Sat}_\alpha(\ulcorner \varphi \urcorner)$ ” denote this formula.

Let Γ be the positive inductive definition used before: If X is a set of $\mathcal{L}_V^+$ sentences, declare that

$$\begin{aligned} \varphi \in \Gamma(X) \quad \text{iff} \quad & \varphi \in X \cup T^+ \text{ or} \\ & \varphi \text{ is } \neg \exists x \varphi, \text{ where } X \cup T^+ \vdash \neg \psi(F(\underline{\alpha})/x), \text{ for all } \alpha < \infty. \end{aligned}$$

Let T^* be the least fixed point of Γ . Then

- $T^* \supseteq T^+$;
- if $T^* \vdash \neg \psi(F(\underline{\alpha})/x)$, for all $\alpha < \infty$, then $T^* \vdash \neg \exists x \psi$;
- T^* is V -amenable and is definable over $\text{Hyp}(V)$, hence $(W; V, T^*) \models \text{ZFC}$; and
- $\widehat{W} \models T^*$.

The reason that $\widehat{W} \models T^*$ is that the set of sentences that hold in $\widehat{W}$ is a fixed point of Γ and T^* is the smallest fixed point of Γ .

Define the forcing $\mathbb{P}$ to consist of all sentences p of $\mathcal{L}_V^+$ such that $T^* + p$ is consistent. Order $\mathbb{P}$ by $p \leq q$ iff $T^* \vdash p \rightarrow q$. Then $\mathbb{P}$ is definable over $(V; T^*)$, hence $(W; V, \mathbb{P}) \models \text{ZFC}$.

When p is a condition, let “ $\text{th}(p)$ ” denote the V -definable class of all $\mathcal{L}_V^+$ consequences of $T^* + p$.

As before, using that T^* is a fixed point of Γ , if $\exists x \psi$ is a sentence of $\mathcal{L}_V^+$, then

$$D_{\exists x \psi} = \left\{ p \in \mathbb{P} : \text{either } \neg \exists x \psi \in \text{th}(p) \text{ or } \varphi(F(\underline{\alpha})/x) \in \text{th}(p), \text{ for some } \alpha < \infty \right\}$$

is dense in $\mathbb{P}$. Just as before, the classes

$$D_{\bigvee \Phi} = \left\{ p \in \mathbb{P} : \text{either } \neg \bigvee \Phi \in \text{th}(p) \text{ or } \varphi \in \text{th}(p), \text{ for some } \varphi \in \Phi \right\}$$

and

$$D_{\neg \varphi} = \left\{ p \in \mathbb{P} : \text{either } \varphi \in \text{th}(p) \text{ or } (\neg \varphi) \in \text{th}(p) \right\}$$

are dense, as well.

As before, call a filter *complete* if it meets all of these dense classes.

Claim. Assume that $G \in \text{StSpc}(\mathbb{P})$ is complete. Then there exists a canonical standard $\mathcal{L}^+$ -structure $\widehat{V}_G = \langle V_G; V, F^{\widehat{V}_G}, S_\alpha^{\widehat{V}_G}, a \rangle_{a \in V, \alpha < \infty}$ such that $\widehat{V}_G$ is eunonymous and

- (1) $\widehat{V}_G \models \varphi$ iff $\varphi \in \text{th}(G) = \bigcup_{p \in G} \text{th}(p)$, for sentences φ of $\mathcal{L}_V^+$; hence, $\widehat{V}_G \models T^+$;
- (2) $V_G \subseteq V[G]$; and
- (3) $V[G] \subseteq V_G$.

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PROOF: (1) is exactly as before.

(2) is also as before, but let us recall the proof to introduce notation that will be needed later. Working in $(V; \mathbb{P})$, define Shoenfield terms $F^*(\alpha)$ by recursion on α as follows:

$$F^*(\alpha) = \left\{ (F^*(\beta), F(\underline{\beta}) \hat{=} F(\underline{\alpha})) : \beta < \alpha \text{ and } T^* + F(\underline{\beta}) \hat{=} F(\underline{\alpha}) \text{ is consistent} \right\}.$$

Then $F^*(\alpha)^{V[G]} = F^{\widehat{V}_G}(\alpha)$.

A new argument is needed for (3). Let $\mathring{a}$ be a Shoenfield term. Then $\mathring{a} \in V \subseteq V_G$. It suffices to see that $G \cap \text{TC}(\{\mathring{a}\}) \in V_G$ because then $\mathring{a}^G$ can be defined by recursion, working in $\widehat{V}_G$. Let $\alpha < \infty$ be such that every $p \in \mathbb{P} \cap \text{TC}(\{\mathring{a}\})$ is a Σ_α sentence. Now $p \in G \cap \text{TC}(\{\mathring{a}\})$ iff $p \in \mathbb{P} \cap \text{TC}(\{\mathring{a}\})$ and $\widehat{V}_G \models p$. But satisfaction of Σ_α sentences is definable using Sat_α in $\widehat{V}_G$. Hence $G \cap \text{TC}(\{\mathring{a}\}) \in V_G$. $\square$

Claim. *If $G \in \text{StSpC}(\mathbb{P})$ is complete, then G is generic over $(V; \mathbb{P})$ with respect to the language of set theory.*

PROOF: As before, it suffices to define a translation procedure $\varphi \mapsto \varphi^*$ from first-order (finitary) sentences of the forcing language into $\mathcal{L}_V^+$ sentences such that

$$(\dagger) \quad V[G] \models \varphi \quad \text{iff} \quad \widehat{V}_G \models \varphi^*.$$

(Again, this suffices because, then, $p \Vdash \varphi$ iff $\text{th}(p) \vdash \varphi^*$, hence a complete filter contains conditions forcing either φ or $\neg\varphi$, for all sentences φ of the forcing language.)

This is done essentially as before:

$$\begin{aligned} (\mathring{a} \hat{=} \mathring{b})^* &= \bigvee_{\mathring{c} \in \text{dom}(\mathring{b})} (\bigvee \mathring{b}(\mathring{c}) \wedge (\mathring{c} \hat{=} \mathring{a})^*) \\ (\mathring{b} \hat{=} \mathring{a})^* &= \bigwedge_{\mathring{c} \in \text{dom}(\mathring{b})} (\bigvee \mathring{b}(\mathring{c}) \rightarrow (\mathring{c} \hat{=} \mathring{a})^*) \\ (\varphi \wedge \psi)^* &= (\varphi^* \wedge \psi^*) \\ (\neg\varphi)^* &= (\neg\varphi^*) \\ (\forall x\varphi)^* &= \forall \delta \text{Sat}_\alpha(\ulcorner \varphi(F^*(\delta)/x) \urcorner^*), \text{ where } \alpha \text{ is least} \\ &\quad \text{such that } \varphi(F^*(\delta)/x)^* \text{ is a } \Sigma_\alpha \text{ formula (for any } \delta) \end{aligned}$$

Using the previous claim, $(\dagger)$ follows by induction on φ . $\square$

To finish the proof, set

$$G = \left\{ p \in \mathbb{P} : \widehat{W} \models p \right\}.$$

Clearly G is complete, hence generic over $(V; \mathbb{P})$. Furthermore, $G \in \text{Hyp}(W; V)$ because $\widehat{W} \in \text{Hyp}(W; V)$. Because both $\widehat{W}$ and $\widehat{V}_G$ are eunonymous and satisfy exactly the sentences of $\text{th}(G)$, we have that $\widehat{W} = \widehat{V}_G$, and so $W = V_G = V[G]$.

Let $\mathcal{F}$ be the set of $(V; \mathbb{P})$ -definable class Shoenfield terms. Then $\mathring{A}$ is definable over $\text{Hyp}(W; V)$, hence $\mathring{A}^G$ is definable over $\text{Hyp}(W; V)$, for each $\mathring{A} \in \mathcal{F}$. It follows that, $(V[G]; \mathring{A}^G)_{\mathring{A} \in \mathcal{F}} \models \text{ZFC}$ because ∞ is definably regular in $\text{Hyp}(W; V)$. In particular, $(V[G]; V, \mathbb{P}, G) \models \text{ZFC}$. $\square$

7. References

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