Cofinalities and approachable ordinals below $\aleph_{\omega+1}$

Assume that $2^{\aleph_\omega} = \aleph_{\omega+1}$. If $\lambda < \aleph_{\omega+1}$ is regular and $S \subseteq \aleph_{\omega+1}$, let $S^{(\lambda)}$ be the set of limit points of $S$ having cofinality $\lambda$. If $\mu < \lambda$, say that $S$ $\lambda$-reflects at cofinality $> \mu$ if there exists a stationary set of $\alpha < \aleph_{\omega+1}$ such that $\text{cf}(\alpha) > \mu$ and $S^{(\lambda)} \cap \alpha$ is stationary in $\alpha$. We show that if $\aleph_\omega$ is a strong limit, then, for $n < \omega$, the set of unapproachable ordinals below $\aleph_{\omega+1}$ does not $\aleph_{n+1}$-reflect at cofinality $> 2^{\aleph_n}$. A consequence is that if $\aleph_\omega$ is a strong limit in $V$ and $W$ is an $\aleph_\omega$ and $\aleph_{\omega+1}$ preserving outer model of $V$ in which the approachable ordinals in the sense of $V$ are non-stationary, then $\{ k < \omega : \text{cf}^W(\aleph^V_k) = \aleph^W_n \}$ is infinite, for all $n > 0$.

We also discuss an unrelated result that if a sufficiently supercompact embedding lifts to an outer model “automatically”, then every “small” set in the outer model is set generic over the inner model.