Purpose: To investigate properties of products of lower triangular matrices.
Prerequisite: Section 2.1
MATLAB functions used: + , * , tril, rand, eye

1. Create several lower triangular matrices, calculate their products in pairs, and see what appears to be true. For example you could do this for two different 2x2 matrices whose lower triangle entries are random numbers by typing:

   \[
   n = 2 \\
   L1 = \text{tril}(\text{rand}(n)), L2 = \text{tril}(\text{rand}(n)), L1*L2, L2*L1
   \]

   Execute the second line several times and inspect the result each time. Repeat with \(n = 3, 4, 5\). The easy way is to press the up arrow on your keyboard to retrieve the second line, and then press [Enter] to execute it again. (Each time you execute \(\text{rand}(n)\) it produces a new \(n \times n\) matrix with random number entries; the function \(\text{tril}\) puts zeros in the upper triangle, so \(\text{tril}(\text{rand}(n))\) produces a random lower triangular matrix.)

   (a) (hand) For each pair \(L_1\) and \(L_2\), what entries of \(L_1L_2\) and \(L_2L_1\) appear to be the same? Is that true for non-triangular matrices? (Try the following calculation several times: \(A = \text{rand}(2), B = \text{rand}(2), A*B, B*A\).)

   (b) (hand, use back or attach an extra sheet) Prove that the product of any two \(n \times n\) lower triangular matrices is lower triangular. Here is one way to begin: "Let \(L_1 = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\) and \(L_2 = \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}\)." Then write out what the \(i,j\) entry of \(L_1L_2\) looks like and explain why it must be zero when \(i < j\).

2. Now investigate products of lower triangular matrices which have all diagonal entries equal to 1. Such a matrix is called a unit lower triangular matrix. For example you could type

   \[
   n = 2 \\
   L1 = \text{tril}(\text{rand}(n), -1) + \text{eye}(n), L2 = \text{tril}(\text{rand}(n), -1) + \text{eye}(n), L1*L2, L2*L1
   \]

   Execute this line several times and inspect the result each time. Repeat with \(n = 3, 4, 5\). (The second input \(-1\) for \(\text{tril}\) causes zeros to be put above the first subdiagonal, that is, on the main diagonal as well as in the upper triangle. So adding the identity matrix \(\text{eye}(n)\) produces a random lower triangular matrix with 1's on the diagonal.)

   (a) (hand) For each pair \(L_1\) and \(L_2\), what entries of \(L_1L_2\) and \(L_2L_1\) appear to be the same?

   (b) (hand, use back or attach an extra sheet) Prove that the product of any two \(n \times n\) unit lower triangular matrices is lower triangular. Notice all you need to prove here is that each diagonal entry of the product is 1 (why?). Begin the same way as in 1(b), but this time let each diagonal entry be 1; then write out what the \(i,i\) entry of \(L_1L_2\) looks like, and explain why it must be 1.

3. (Extra Credit) Suppose \(L\) is an \(n \times n\) lower triangular matrix with each diagonal entry nonzero. Create \(A = [L \ I]\), where \(I\) denotes the \(n \times n\) identity matrix. Explain why the reduced echelon form of \(A\) must be of the form \([I \ K]\), where \(K\) is another \(n \times n\) lower triangular matrix with nonzero diagonal entries.