Computer Project: Schur Complements  

Name

Purpose: To learn what Schur complements are and their connection with row reduction.

Prerequisite: Section 2.4

MATLAB functions used: inv, eye, zeros, -, *, and schurdat from Lay's Toolbox

Background. This is based on Exercise 15 in Section 2.4. Let \( A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \) be a partitioned matrix in which \( A_{11} \) is square and invertible. Define the Schur complement of \( A_{11} \) in \( A \) to be \( S = A_{22} - A_{21}A_{11}^{-1}A_{12} \). Here you will see three ways to calculate the matrix \( \begin{bmatrix} A_{11} & A_{12} \\ O & S \end{bmatrix} \), where \( O \) is the zero matrix having the same shape as \( A_{21} \).

1. (MATLAB) Type schurdat to get \( A_{11} = \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}, A_{12} = \begin{bmatrix} 4 & 1 & -1 \\ -2 & 0 & 3 \end{bmatrix}, A_{21} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}, \) and \( A_{22} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \). In MATLAB they will be stored as \( A_{11}, A_{12}, A_{21}, A_{22} \).

   (a) Type \( A = [A_{11} \ A_{12}; A_{21} \ A_{22}] \) to create \( A \). Inspect to see that this does look like \( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \), and use dotted lines to mark the partitions in \( A \):

   \[
   A = \begin{bmatrix} \end{bmatrix}
   \]

   (b) One way to get the Schur complement \( S \) for the matrix here would be to just calculate it directly, using the definition above. Type the following line to do that, and record the result:

   \[
   S = A_{22} - A_{21} \text{inv}(A_{11}) \ast A_{12}
   \]

   \[
   S = \begin{bmatrix} \end{bmatrix}
   \]

2. (hand) Assume now that \( A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \) is any partitioned matrix in which \( A_{11} \) is square and invertible. Let \( L, I \) and \( O \) be of appropriate sizes so that \( \begin{bmatrix} I & O \\ L & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \) is defined. Find a formula for \( L \), in terms of the \( A_{ij} \)'s, so that

   \[
   \begin{bmatrix} I & O \\ L & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ O & S \end{bmatrix}
   \]

   is true; show work:
3. (MATLAB) A second way to get the Schur complement $S$ of $A_{11}$ in $A$ to would be to use your formula from question 2 to calculate $L$, then create $C = \begin{bmatrix} I & O \\ L & I \end{bmatrix}$, and then calculate $CA$. Do this for the matrix in question 1. You figure out a command to calculate $L$ and record your command below. A way is shown to create $C$ and $CA$:

$$L = \quad C = \begin{bmatrix} \text{eye(2)} & \text{zeros(2,2)}; \text{L, eye(2)} \end{bmatrix}$$

Inspect $CA$ to verify that it does look like $\begin{bmatrix} A_{11} & A_{12} \\ O & S \end{bmatrix}$, where $S$ is the same matrix as you got in questions 1 and 2. Record results.

$$L = \quad C = \quad CA =$$

4. (hand) A third way to calculate the Schur complement $S$ of $A_{11}$ in $A$ is to use row operations in a special way. This method actually does the least arithmetic so is the most efficient method. The idea is: don’t change anything in $[A_{11} \ A_{12}]$ but just add multiples of appropriate rows of $[A_{11} \ A_{12}]$ to rows of $[A_{21} \ A_{22}]$ so as to create a block of zeros below $A_{11}$. Notice this is not the usual Row Reduction Algorithm because nothing will change in the top block $[A_{11} \ A_{12}]$.

Verify that this method works for the matrix $A$ from question 1. The first step is shown; you finish. Record each matrix you create and inspect your final matrix to be sure it does look like $\begin{bmatrix} A_{11} & A_{12} \\ O & S \end{bmatrix}$:

$$A = \begin{bmatrix} 0 & -1 & 4 & 1 & -1 \\ 1 & 3 & -2 & 0 & 3 \\ 2 & 0 & 1 & 2 & 3 \\ -1 & 1 & 4 & 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -1 & 4 & 1 & -1 \\ 1 & 3 & -2 & 0 & 3 \\ 0 & -6 & 5 & 2 & -3 \\ -1 & 1 & 4 & 5 & 6 \end{bmatrix}$$

5. (hand) Assume now that $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is any partitioned matrix in which $A_{11}$ is invertible. Prove that the method described in question 5 will always work, to get a block of zeros below $A_{11}$. That is, if $A_{11}$ is invertible, you can always do row operations to $A$ as described in question 5, to get the form $\begin{bmatrix} A_{11} & A_{12} \\ O & W \end{bmatrix}$. Also explain why the block $W$ obtained this way must be the Schur complement of $A_{11}$ in $A$. Attach an extra sheet. (Hint: you must use the invertibility of $A_{11}$ somehow!)