1. (8 points) Let $s_n$ be a bounded sequence, and let $S_k = \{s_n \mid n \geq k\}$. Using this notation, define $\lim \sup s_n$.

2. (12 points) Determine if the series $\sum_{n=1}^{\infty} \frac{3^{1/n} \sqrt{n}}{2n^2 - 5}$ converges or diverges, and prove your answer. If you use a test, cite it by name; if the test involves a parameter, state the value of the parameter and the condition it satisfies.

In questions 3–5, you are given a statement. If the statement is true, you need only write “True”, though a justification may earn you partial credit if the correct answer is “False”. If the statement is false, write “False”, and justify your answer as specifically as possible. (Do not just write “T” or “F”, as you may not receive any credit; write out the entire word “True” or “False”.)

3. (12 points) TRUE/FALSE: Let $\sum a_n$ be a series with $a_n \geq 0$ for all $n$. If $\sum a_n$ converges, then $\sum a_n^2$ must also converge.

4. (12 points) DELETED: comes from Ch. 18.

5. (12 points) TRUE/FALSE: Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. It is possible that $g(\sqrt{2} + \frac{1}{n}) = 7 + 3^{-n}$ for all $n \in \mathbb{N}$ and $g(\sqrt{2}) = 4$.

6. (14 points) PROOF QUESTION. Let $s_n$ be a sequence such that $s_1 = 3$ and $s_n < s_{n+1} < \frac{s_n + 7}{2}$ for all $n \in \mathbb{N}$.

(a) Use induction to prove that $s_n < 7$ for all $n \in \mathbb{N}$. (If you can’t prove this, assume it and go on to the rest of the problem.)

(b) Prove that $\lim s_n$ exists.

7. (15 points) PROOF QUESTION. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} 
0 & \text{if } x \text{ is rational,} \\
x \sin \left( \frac{1}{x} \right) & \text{if } x \text{ is irrational.}
\end{cases}$$

Prove that $h$ is continuous at 0.

8. (15 points) PROOF QUESTION. Let $s_n$ be a sequence such that $-5 \leq s_n \leq 22$ for all $n \in \mathbb{N}$. Prove that there exists a sequence of positive integers $n_k \in \mathbb{N}$ with three properties:

- $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$ (i.e., $n_k < n_{k+1}$ for all $k \in \mathbb{N}$);
- $\lim_{k \to \infty} s_{n_k}$ exists; and
- $-5 \leq \lim_{k \to \infty} s_{n_k} \leq 22$. 