On a result of Littlewood concerning prime numbers

by

D. A. Goldston (Berkeley, Calif.)

1. Introduction. We define

\[ \psi(x) = \sum_{n \leq x} A(n) \]

where

\[ A(n) = \begin{cases} \log p, & n = p^m, p \text{ prime, } m \text{ integer } \geq 1, \\ 0 & \text{otherwise}. \end{cases} \]

The prime number theorem is equivalent to

\[ \psi(x) \sim x \quad (\text{as } x \to \infty). \]

Assuming the Riemann Hypothesis (the RH), we have the more precise result

\[ \psi(x) - x = O(x^{1/2} \log x) \]

and, on the other hand, we have (without hypothesis)

\[ \psi(x) - x = \Omega_\pm (x^{1/2} \log \log \log x). \]

The result (1.4) is due to von Koch in 1901, while (1.5) was proved by Littlewood in 1914 (see [4], Chapters 4, 5). Presumably (1.5) is nearer to the truth. The basis for these results is the explicit formula for \( \psi(x) \):

\[ \frac{\psi(x+0) + \psi(x-0)}{2} = x - \sum_{\eta} \frac{x^\eta}{\eta} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2} \log (1-x^{-2}) \]

the summation being over the non-trivial zeros of the zeta function, \( \eta = \beta + i\gamma \). (The RH allows us to take \( \beta = 1/2 \).) The series in (1.6) is neither absolutely nor uniformly convergent, and is understood as

\[ \sum_{\eta} \frac{x^\eta}{\eta} = \lim_{x \to \infty} \sum_{|\eta| < T} \frac{x^\eta}{\eta}. \]
For applications it is often useful to replace (1.6) by a formula due to Landau ([6], pp. 108–120, [1], Ch. 5): For \( k \) some absolute positive constant,

\[
(1.7) \quad |\psi(x) - x + \sum_{\nu < \sqrt{y}} \frac{x^\nu}{\nu} < k \left( \frac{x \log^2 x}{y} + \frac{x \log y}{y} + \log x \right) \quad (x \geq 3, y \geq 3).
\]

If \( y \geq x^{1/3} \log x \), (1.7) implies \( (k^\prime \) an absolute constant),

\[
(1.8) \quad |\psi(x) - x + \sum_{\nu < \sqrt{y}} \frac{x^\nu}{\nu} < k^\prime x^{1/3} \log x \quad (x \geq 5, y \geq x^{1/3} \log x).
\]

Assuming the Riemann Hypothesis, Littlewood proved in [7] that the condition \( y \geq x^{1/3} \log x \) in (1.8) can be replaced by \( y \geq x^{1/3} \). In this paper we show, again assuming the Riemann Hypothesis, that we can take \( y \geq x^{1/3} \log x \).

This slight improvement allows us to give a simple proof of a result due to Cramér in 1919; assuming the RH, and letting \( p_n \) denote the \( n \)th prime,

\[
(1.9) \quad p_{n+1} - p_n = O(p_n^{1/2} \log p_n)
\]

(see [2], [3]). Our proof is similar in principle to the proof given by Ingham ([5], p. 256), but proceeds more directly to the result. We also give a simple proof of the closely related result, assuming the RH, \( h \leq x \),

\[
(1.10) \quad \pi(x+h) - \pi(x) \sim \frac{h}{\log x}, \quad \frac{h}{x^{1/3} \log x} \to \infty \quad \text{as} \quad x \to \infty.
\]

Here \( \pi(x) \) is the number of primes less than or equal to \( x \). This result was stated by Selberg [9].

2. A lemma. We need a lemma due to Littlewood [7].

**Lemma.** If \( |z| \leq 1/2, \; |ms| \leq 2 \), then

\[
|(1+z)^m - 1 - ms| \leq 2.6 |m|(|m|+1)|z|^2.
\]

**Proof.** Let \( |z| = r, \; |m| = \mu \), and we may suppose \( r > 0, \; \mu > 0 \).

We have

\[
T = \left| \frac{(1+z)^m - 1 - ms}{|m|(|m|+1)|z|^2} \right| \leq \sum_{n=1}^{\infty} \frac{|m|(|m|-1) \cdots (m-n+1)|z|^n}{\mu(\mu+1)n!} r^{n-1}
\]

\[
\leq \sum_{n=1}^{\infty} \frac{\mu(\mu+1) \cdots (\mu+n-1)}{\mu(\mu+1)n!} r^{n-1} = \frac{(1-r)^{-\mu} - 1 - \mu r}{\mu(\mu+1)r^2}.
\]
With \( r \) fixed, the second to last expression clearly increases with \( \mu \), and so is maximum when \( \mu = 2/r \). Thus we have
\[
T \leq \frac{(1-r)^{-2r} - 3}{2(2+r)}.
\]
This last expression is strictly increasing for \( 0 < r < 1 \). To see this, we differentiate and obtain
\[
\frac{1}{2(2+r)^2(1-r)^{2r}} [(2r^{-2} \log(1-r) + 2r^{-1}(1-r)^{-1})(2+r) - 1] + \frac{3}{2(2+r)^3}.
\]
Expanding \( \log(1-r) \) and \( (1-r)^{-1} \) into power series (valid for \( 0 < r < 1 \)) and multiplying out shows this expression is positive. Since \( 0 < r \leq 1/2 \), we conclude
\[
T \leq \frac{(1-0.5)^{-4} - 3}{2(2.5)} = \frac{16 - 3}{5} = 2.6.
\]

3. The main theorem. Throughout the rest of this paper we will assume the Riemann Hypothesis. Thus the complex zeros of \( \xi(x) \) are \( \rho = \beta + iy = \frac{1}{2} + iy \). We denote by \( \theta \) a number satisfying \( |\theta| = 1 \). The number denoted will, in general, be different for different occurrences and may depend on variables. Most of our formulas will hold "for \( x \) sufficiently large", and we will denote this by "\( x > A \)" where \( A \) is some positive absolute constant which may differ on different occasions.

**Theorem 1.** Assuming the Riemann Hypothesis, we have
\[
|\psi(x) - x + \sum_{|\rho| < \sqrt{y}} \frac{x^\rho}{\rho} | < \frac{x}{2y} + 2x^{1/2} \log y, \quad x \geq 3, \quad y > A.
\]
In particular,
\[
|\psi(x) - x | = - \sum_{|\rho| < \sqrt{y}} \frac{x^\rho}{\rho} + O(x^{1/2} \log x)
\]
uniformly for \( y \geq x^{1/2} \log x, \quad x > A \); and
\[
|\psi(x) - x + \sum_{|\rho| < x^{1/2} \log x} \frac{x^\rho}{\rho} | < 1.5x^{1/2} \log x, \quad x > A.
\]

**Proof.** Let \( \psi_1(x) = \int_0^x \psi(r) \, dr \). The explicit formula for \( \psi_1(x) \) is, for \( x \geq 1 \),
\[
\psi_1(x) = \frac{x^2}{2} - \sum_{\rho} \frac{x^{\rho+1}}{\rho (\rho+1)} - 2 \frac{x}{\zeta(0)} + \frac{x}{\zeta(-1)} - \sum_{r=1}^{\infty} \frac{x^{1-2r}}{2r (2r-1)}.
\]
(see [4], p. 73). Let $h$ be a function of $x$ such that $1 \leq h \leq x/2$. Then
\[
\frac{\psi_1(x \pm h) - \psi_1(x)}{\pm h} = x \pm \frac{h}{2} - \sum_{e \neq 0} \frac{(x \pm h)^{e+1} - x^{e+1}}{e(e+1)(\pm h)} + \frac{\zeta'}{\zeta}(0) \pm \frac{1}{h} \sum_{r=1}^{\infty} \frac{(x \pm h)^{1-2r} - x^{1-2r}}{2r(2r-1)}.
\]
Now $\frac{\zeta'}{\zeta}(0) = \log 2\pi < 2$; and for $x \geq 3$,
\[
\left| \pm \frac{1}{h} \sum_{r=1}^{\infty} \frac{(x \pm h)^{1-2r} - x^{1-2r}}{2r(2r-1)} \right| \leq \sum_{r=1}^{\infty} \frac{x^{-1}(2+1)}{2r(2r-1)} \leq \sum_{r=1}^{\infty} \left[ \frac{-1}{2r} + \frac{1}{2r-1} \right] = 1.
\]
Hence for $x \geq 3$, $1 \leq h \leq x/2$,
\[
\psi_1(x \pm h) - \psi_1(x) = x \pm \frac{h}{2} - \sum_{|r| < \infty} \frac{(x \pm h)^{e+1} - x^{e+1}}{e(e+1)(\pm h)} - \sum_{|r| > \infty} \frac{(x \pm h)^{e+1} - x^{e+1}}{e(e+1)(\pm h)} + K,
\]
where $K$ depends on $x$ and $h$, and $|K| < 3$.

We have (without hypothesis)
\[
\sum_{T < \gamma^3} \frac{1}{\gamma^3} = \frac{1}{2\pi} \frac{\log T}{T} + O\left(\frac{1}{T}\right) \text{ as } T \to \infty
\]
(see [4], Th. 25b; an argument like the one on p. 98 gives this result).

The second sum on the right of (3.5) is in absolute value
\[
< \sum_{|r| > \infty} \frac{((\frac{1}{2})^{3/2} + 1)x^{3/2}}{\gamma^2 h} < \frac{6x^{3/2}}{h} \sum_{y > 0} \frac{1}{\gamma^2} = \frac{6x^{3/2}}{h} \frac{2\pi y}{2\pi y} (1 + o(1)) < \frac{x^{3/2}\log y}{h y},
\]
for $y > A$.

We have used here $h \leq x/2$, (3.6), and the fact that the zeros of $\zeta(s)$ are symmetric with the real axis. Next, the first sum in (3.5) is equal to
\[
- \sum_{|r| < \infty} \frac{x^e}{e} - \sum_{|r| < \infty} \frac{(x \pm h)^{e+1} - x^{e+1}}{e(e+1)(\pm h)}.
\]
Denote by $w_e$ the general term of the second sum. Thus
\[
w_e = x^{e+1} \frac{(1 \pm h/x)^{e+1} - 1 \mp h/x}{e(e+1)(\pm h)}.
\]
We now apply the lemma taking \( z = \pm h/x \), \( m = \varepsilon + 1 \), and impose the condition

(3.7) \[ y \leq x/h. \]

The two conditions of the lemma are thus satisfied, for \( |z| = h/x \leq 1/2 \), and, since \( |\gamma| < y \) in our sum,

\[ |mx| = |\varepsilon + 1|(h/x) \leq (3/2 + |\gamma|)(h/x) \leq (3/2 + y)(h/x) \leq 3/4 + 1 < 2. \]

Therefore,

\[ \left| w_{\varepsilon} \right| \leq \frac{2.6 \cdot |\varepsilon + 1|(|\varepsilon + 1| + 1)(h/x)^2}{|\varepsilon|(|\varepsilon + 1|)(h/x)} \]
\[ = 2.6x^{-1/2} h \frac{|\varepsilon + 1| + 1}{|\varepsilon|} \leq 2.6x^{-1/2} h(1 + (2/|\varepsilon|)) \]
\[ < 2.6x^{-1/2} h(1 + 1/7) < 3x^{-1/2} h, \]

since \( 1/|\varepsilon| < 1/|\gamma| < 1/14 \).

Let \( N(T) \) denote the number of zeros of \( \zeta(s) \) with \( 0 < \gamma < T \). Then (without hypothesis)

(3.8) \[ N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T), \]

and consequently

(3.9) \[ N(T) < \frac{T}{2\pi} \log T \text{ for } T > A. \]

Returning to our sum,

\[ \left| - \sum_{|\gamma| < y} w_{\varepsilon} \right| \leq 6hx^{-1/2} \sum_{0 < y < y} 1 < 6hx^{-1/2} \left( \frac{y}{2\pi} \log y \right) < x^{-1/2} hy \log y, \]

for \( y > A \). Combining these results in (3.5) we obtain

(3.10) \[ \psi_1(x \pm h) - \psi_1(x) = \frac{x \pm h}{2} + \theta \frac{x^{3/2} \log y}{hy} + \theta_2 x^{-1/2} hy \log y - \sum_{|\gamma| < y} \frac{x^\varepsilon}{\varepsilon} \]

for \( y > A \), \( x \geq 3 \), \( 1 \leq h \leq x/2 \), and subject to (3.7). (The term \( K \) was absorbed into \( \theta, x^{3/2} \log y/hy \) since we rounded up to obtain this estimate, and by (3.7) this term is \( \geq 6x^{1/2} \)). The \( \theta \)'s depend on \( x \) and \( h \), and will be different in the cases \( +h \) and \( -h \).

Since \( \psi(x) \) is nondecreasing, we have

\[ \frac{\psi_1(x + h) - \psi_1(x)}{-h} = \frac{1}{h} \int_{x-h}^{x} \psi(\tau) d\tau \leq \psi(x) \leq \frac{1}{h} \int_{x}^{x+h} \psi(\tau) d\tau = \frac{\psi_1(x + h) - \psi_1(x)}{h}. \]
Hence from (3.10) we obtain, subject to the same conditions,

\[(3.11) \quad \left| \psi(x) - x + \sum_{|\tau| < \sqrt{x}} \frac{x^\tau}{\tau} \right| < \frac{x^{1/2} \log y}{hy} + x^{-1/2} \log \log y + \frac{\lambda}{2} \cdot \]

Comparing the first two terms on the right, we choose \(hy = x\). Thus (3.7) is satisfied, and we have, for \(y > A\), \(x \geq 3\),

\[\left| \psi(x) - x + \sum_{|\tau| < \sqrt{x}} \frac{x^\tau}{\tau} \right| < \frac{x}{2y} + 2x^{1/2} \log y.\]

This proves (3.1).

We now pick \(x^{1/2} \log x < y < x\) and obtain

\[\left| \psi(x) - x + \sum_{|\tau| < \sqrt{x}} \frac{x^\tau}{\tau} \right| < \frac{1}{2} x^{1/2} \log x + 2x^{1/2} \log x = O(x^{1/2} \log x).\]

For \(y \geq x\) Landau's result (1.7) implies

\[\psi(x) - x + \sum_{|\tau| < \sqrt{x}} \frac{x^\tau}{\tau} = O(\log^2 x) \quad (y \geq x \geq 3).\]

Equation (3.2) now follows.

Finally, setting \(y = x^{1/2} / \log x\) in (3.1), we have, for \(x > A\),

\[\left| \psi(x) - x + \sum_{|\tau| < x^{1/2} / \log x} \frac{x^\tau}{\tau} \right| < \frac{1}{2} x^{1/2} \log x + 2x^{1/2} \log x^{1/2} - 2x^{1/2} \log \log x < \frac{3}{2} x^{1/2} \log x.\]

4. Application to Cramér’s theorem. As a simple consequence of our theorem we have

**Theorem 2** (Cramér). *Assuming the Riemann Hypothesis, we have*

\[(4.1) \quad \pi(x + 5x^{1/2} \log x) - \pi(x) > x^{1/2} \quad \text{for } x > A,\]

*and*

\[(4.2) \quad P_{n+1} - p_n < 4p_n^{1/2} \log p_n \quad \text{for } n > A.\]

**Proof.** In what follows we suppose \(x\) is sufficiently large, and will not indicate it again.

Let \(1 \leq h \leq x/5\). Replacing \(x\) by \(x + h\) in (3.1) and taking \(y = x^{1/2} / \log x\), we have

\[\left| \psi(x + h) - (x + h) + \sum_{|\tau| < x^{1/2} / \log x} \frac{(x + h)^\tau}{\tau} \right| < \frac{(x + h) \log x}{2x^{1/2}} + 2(x + h)^{1/2} \log (x^{1/2})\]

\[< x^{1/2} \log x + [\frac{1}{2}] x^{1/2} \log x < 1.7 x^{1/2} \log x.\]
Combining this with (3.3) we have

\[(4.3) \quad \psi(x+h) - \psi(x) = h - \sum_{n \leq x^{1/3} \log x} \frac{(x+h)^n - x^n}{e} + 3.2 \theta x^{1/3} \log x.\]

Since

\[\left| \frac{(x+h)^n - x^n}{e} \right| = \left| \int_x^{x+h} x^{t-1} dt \right| \leq x^{-1/2} h,\]

\[(4.4) \quad \psi(x+h) - \psi(x) = h + 6x^{-1/2} h \sum_{n \leq x^{1/3} \log x} 1 + 3.2 \theta x^{1/3} \log x\]

\[= h + 6x^{-1/2} h \left[ \frac{1}{2\pi} \frac{x^{1/2}}{\log x} \log x^{1/3} \log x \right] + 3.2 \theta x^{1/3} \log x,\]

by (3.9),

\[= h + \frac{6h}{2\pi} + 3.2 \theta x^{1/3} \log x.\]

Thus, taking \( h = 5x^{1/3} \log x, \) we have

\[(4.5) \quad \psi(x+5x^{1/3} \log x) - \psi(x) > 5x^{1/3} \log x - \left( \frac{5}{2\pi} + 3.2 \right) x^{1/3} \log x > x^{1/3} \log x.\]

Finally, we have for \( 1 \leq h \leq x \) (\([w] = \text{integer part of } x\)),

\[(4.6) \quad \psi(x+h) - \psi(x) = \sum_{x<p<x+h} \log p + O \left( \sum_{2<p<x+h} \log p \left[ \frac{\log(x+h)}{\log p} \right] \right),\]

\[= \sum_{x<p<x+h} (\log x + O(1)) + O \left( \sum_{p<x} \log 2x \right)\]

\[= \{\pi(x+h) - \pi(x)\} (\log x + O(1)) + O(x^{1/2}).\]

Combining (4.5) and (4.6) gives (4.1). Next, taking \( h = 4x^{1/3} \log x \) in (4.4) gives

\[(4.7) \quad \psi(x+4x^{1/3} \log x) - \psi(x) > 4x^{1/3} \log x > 0.\]

Equation (4.6) now implies \( \pi(x+4x^{1/3} \log x) - \pi(x) > 4x > 0.\) Taking \( x = p_n, \) we see \( p_{n+1} - p_n < 4x^{1/3} \log x = 4p_n^{1/2} \log p_n.\)

The constants in (4.1) and (4.2) can be decreased. The 5 in (4.1) may be replaced by a number less than 4 and the 4 in (4.2) by a number less than 2. It is interesting to compare this with the conjectured result ([18])

\[(4.8) \quad \psi(x+h) - \psi(x) = h + O(h^{1/2} x), \quad 1 \leq h \leq x.\]
We can give an easy proof of the best result known in this direction, assuming the RH. It is stated in [9].

**Theorem 3.** Assume the Riemann Hypothesis. Let \( h \) be a function of \( x \) such that (i) \( h \leq x \), (ii) \( h \) is monotonically increasing, and (iii) \( h/(x^{1/2} \log x) \to \infty \) as \( x \to \infty \). Then

\[
\psi(x + h) - \psi(x) \sim h
\]

and

\[
\pi(x + h) - \pi(x) \sim h/\log x.
\]

**Proof.** The two assertions are equivalent by (4.6) and (iii). Thus we shall prove (4.9). Let \( \varphi(x) \) be any function such that \( \varphi(x) \to \infty \) as \( x \to \infty \) and \( \varphi(x) = O(\log x) \). Then by (4.3)

\[
\psi(x + h) - \psi(x) = h - \sum_1 \frac{(x + h)^q - x^q}{q} - \sum_2 \frac{(x + h)^q - x^q}{q} + O(x^{3/2} \log x),
\]

where \( \sum_1 \) is summed over \( |\gamma| < x^{3/2}/(\log x) \varphi(x) \), and \( \sum_2 \) is summed over \( x^{3/2}/(\log x) \varphi(x) \leq |\gamma| < x^{3/2}/\log x \). Handling \( \sum_1 \) as before,

\[
\psi(x + h) - \psi(x) = h + O \left( x^{-1/2} \left( \frac{x^{1/2}}{\log x \varphi(x)} \left( \log \left( \frac{x^{1/2}}{\log x \varphi(x)} \right) \right) \right) \right) + \\
+ O \left( x^{1/2} \sum_2 \frac{1}{\gamma} \right) + O(x^{3/2} \log x).
\]

Since (see [4], p. 98)

\[
\sum_{0 < \gamma < x} \frac{1}{\gamma} = \frac{1}{4\pi} \log^2 T + O(\log T),
\]

we obtain

\[
\psi(x + h) - \psi(x) = h + O \left( \frac{h}{\varphi(x)} \right) + \\
+ O \left( x^{1/2} \left( \frac{1}{4\pi} (\log x^{1/2} - \log \log x)^2 - \frac{1}{4\pi} (\log x^{1/2} - \log \log x - \log \varphi(x))^2 + \\
\right) \right) + O(x^{3/2} \log x)
\]

\[
= h + O \left( \frac{h}{\varphi(x)} \right) + O(x^{1/2} (\log x) (\log \varphi(x))) + O(x^{3/2} \log x).
\]

Hence,

\[
\psi(x + h) - \psi(x) = h + O \left( \frac{h}{\varphi(x)} \right) + O(x^{1/2} (\log x) (\log \varphi(x)))).
\]
We obtain the theorem by picking $h$ larger than the last order term, i.e. $h \geq x^{1/2}(\log x)\varphi(x)$.

We note that by (4.4), $K$ any positive constant, and $x > A$,

$$
(4.13) \quad [K(1-1/2\pi) - 3.2]x^{1/2}\log x < \psi(x + Kx^{1/2}\log x) - \psi(x)
$$

$$
< [K(1+1/2\pi) + 3.2]x^{1/2}\log x.
$$

It seems to require new ideas to replace (4.13) by an asymptotic result. The above proof shows how Theorem 1 must be improved in order to obtain new results on primes in short intervals. Let $\varphi(x)$ be any function monotonically increasing to infinity. Then the result

$$
(4.14) \quad \varphi(x) - x = - \sum_{\psi(y) < x} \frac{x^{\gamma}}{\log x} + O(\psi(x)) \quad \text{uniformly for } y \geq x^{1/2} \frac{\log x}{\log \varphi(x)},
$$

$x > A$, implies (with RH) $\varphi(x + h) - \varphi(x) = h + O(h/\varphi(x)) + O(\psi(x))$, $1 \leq h < x$, $x > A$. This gives (i) $\varphi(x + h) - \varphi(x) \sim h$ if $h/\psi(x) \rightarrow \infty$ as $x \rightarrow \infty$ and (ii) $p_{n+1} - p_n = O(\psi(p_n))$. When $\gamma \sim x^{1/2}/\log x$ the terms in the sum in (4.14) are $O(\log x)$. This, together with the cancellation between terms in the sum makes it seem reasonable that $\psi(x)$ is smaller than in Theorem 1.

References


DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
Berkeley, California 94720, USA

Received on 2.3.1979
and in revised form on 10.11.1979

(1139)