1. (13 points) Let $R$ be a ring (not necessarily commutative).

(a) Define what it means for $R$ to be a ring with unity.

(b) If we now assume that $R$ is a ring with unity, define what it means for $a \in R$ to be a unit.

2. (17 points) Let $G$ be a group whose operation is written multiplicatively, and let $N$ be a normal subgroup of $G$. Define the quotient group (factor group) $G/H$. In particular, describe the elements of $G/H$, the operation in $G/H$, and the identity element of $G/H$.

3. (12 points) Let

$$\alpha = (2 \ 5 \ 3)(4 \ 8 \ 11 \ 9 \ 6)(7 \ 10),$$
$$\beta = (1 \ 10 \ 3 \ 4 \ 8 \ 11 \ 2 \ 7).$$

(a) Calculate $\alpha \beta$. Put your final answer in cycle form, and show all your work.

(b) Calculate $|\alpha \beta|$. Show all your work.

(c) Without further calculation, determine the order of $\beta^{-1} \alpha^{-1}$. Briefly justify your answer.

4. (12 points)

(a) How many abelian groups of order $360 = 2^3 \cdot 3^2 \cdot 5$ are there, up to isomorphism? Briefly justify your answer. (You do not need to list all such groups; just explain how you can be sure that your count is correct.)

(b) Give an example of an abelian group $G$ of order 360 and an element $a \in G$ such that the order of $a$ is 20. Briefly justify your answer.

For questions 5–10, you are given a statement. If the statement is true, you need only write “True”, though a justification may earn you partial credit if the correct answer is “False”. If the statement is false, write “False”, and justify your answer as specifically as possible. (Do not just write “T” or “F”, as you may not receive any credit; write out the entire word “True” or “False”.)

5. (13 points) (TRUE/FALSE) Let $R$ be an integral domain. Then it must be the case that every nonzero element of $R$ is a unit.

6. (13 points) (TRUE/FALSE) It is possible that there exists an abelian group $G$ and a surjective (onto) homomorphism $\varphi : G \to S_4$.

7. (13 points) (TRUE/FALSE) Let $G$ be a group. If there exists an element $a \in G$ such that $\langle a \rangle = G$, and $H$ is a subgroup of $G$, then it must be the case that $H$ is cyclic.

8. (13 points) (TRUE/FALSE) Let $G$ be a group of order 75. It is possible that there exists a subgroup $H$ of $G$ such that $H$ is isomorphic to $U(10)$. 

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9. (13 points) **(TRUE/FALSE)** Let $G$ be a group of order 12 that is not cyclic. Then it must be the case that $G$ is isomorphic to $D_6$.

10. (13 points) **(TRUE/FALSE)** Let $R$ be a commutative ring with unity. Then it must be the case that there are exactly two elements $a \in R$ such that $a^2 = 2a$.

11. (17 points) **PROOF QUESTION.** Let $G$ be an abelian group whose operation is written multiplicatively. Note that the two parts of this question are connected.
   
   (a) Let $\varphi : G \to G$ be defined by $\varphi(g) = g^4$. Prove that $\varphi$ is a homomorphism.
   
   (b) Let $K$ be the set of all elements of $G$ whose order divides 4, or in other words, let
   
   $$K = \{ g \in G \mid |g| \text{ divides } 4 \}.$$
   
   You may take it as given (i.e., do not try to prove) that $K$ is a subgroup of $G$. Prove that there exists a subgroup $H$ of $G$ such that $G/K \cong H$ (i.e., prove that $G/K$ is isomorphic to a subgroup of $G$).

12. (17 points) **PROOF QUESTION.** Let $R$ be a commutative ring such that for all $r \in R$, $r + r = 0$, and let $A$ be an ideal in $R$. Prove that for all $\bar{x}, \bar{y} \in R/A$, we have that $(\bar{x} + \bar{y})^2 = \bar{x}^2 + \bar{y}^2$.

13. (17 points) **PROOF QUESTION.** Recall that $GL(2,R)$ is the group of all invertible $2 \times 2$ matrices with real entries.
   
   Let $X$ be an arbitrary but fixed $2 \times 2$ matrix with real entries, and define
   
   $$H(X) = \{ A \in GL(2,R) \mid AX = X \}.$$
   
   Prove that $H(X)$ is a subgroup of $GL(2,R)$.

14. (17 points) **PROOF QUESTION.** Let $G$ be a group of order $91 = 7 \cdot 13$. Prove that $G$ contains at least one element of order 7.