Sample final, Spring 2002

1. Let \( \mathbf{u}_1, \ldots, \mathbf{u}_k \) be vectors in a subspace \( V \) of \( \mathbb{R}^n \).
   (a) (10 points) Define what it means for \( \mathbf{u}_1, \ldots, \mathbf{u}_k \) to be linearly independent.
   (b) (10 points) Define what it means for \( \mathbf{u}_1, \ldots, \mathbf{u}_k \) to be a basis for \( V \), and define the dimension of \( V \).

2. (14 points) Let \( A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -3 & -4 & -2 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} \).
   (a) Exactly one of the products \( AB, BA \) is defined. Calculate that product.
   (b) Calculate \( A^{-1} \), if it exists, or explain how you know that it does not exist.

3. (14 points) Find the solution set to
   \[
   \begin{align*}
   2x_2 + 2x_4 &= 4, \\
   -x_1 - 2x_2 + x_3 &= 2, \\
   2x_1 + x_2 - 2x_3 &= -1.
   \end{align*}
   \]
   If the system is consistent, put your final answer in vector form; if the system is not consistent, explain how you know that the system is not consistent.

4. (20 points) Let \( A = \begin{bmatrix} 1 & 2 & -4 & 1 & 0 \\ 0 & 1 & -3 & 1 & 1 \\ 3 & 2 & 0 & 0 & -3 \\ -1 & 1 & -5 & 1 & 2 \end{bmatrix} \), \( \text{ref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \).
   (a) Find a basis for the column space of \( A \).
   (b) Find the dimension of the nullspace of \( A \).
   (c) Find one specific nonzero vector \( \mathbf{x} \) such that \( A\mathbf{x} = \mathbf{0} \).
   Show all your work, and in each part this question, briefly EXPLAIN (in a phrase or sentence) how your answer was obtained.

5. (14 points) Let \( W = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix} \right\} \). Find an orthogonal basis for \( W \).
   Show all your work.

6. (20 points) Let \( A = \begin{bmatrix} -1 & -3 & -6 \\ -6 & -4 & -12 \\ 3 & 3 & 8 \end{bmatrix} \). A computation shows that the characteristic polynomial of \( A \) is \( -(t + 1)(t - 2)^2 \). (This is given; do not spend time checking it.)
   Determine if \( A \) is diagonalizable. If \( A \) is diagonalizable, find a diagonal matrix \( D \) and an invertible matrix \( P \) such that \( A = PDP^{-1} \); if \( A \) is not diagonalizable, explain how you know that \( A \) is not diagonalizable.
7. (T/F) (8 points) Let $A$ be a $4 \times 4$ matrix such that 3 is an eigenvalue of $A$. It is possible that the nullspace of $(A - 3I_4)$ is the zero subspace.

8. (T/F) (8 points) Let $V$ be a 2-dimensional subspace of $\mathbb{R}^4$, and let $x, y, z$ be nonzero vectors in $V$. Then $\{x, y, z\}$ must be linearly independent.

9. (T/F) (8 points) There exist $3 \times 3$ matrices $A$ and $B$ such that $\det A = 2$, $\det B = -4$, and $AB$ is not invertible.

10. (T/F) (8 points) Let $W$ be a nonzero subspace of $\mathbb{R}^5$. If $v$ is a vector in $\mathbb{R}^5$, and $v = x + y$, where $x \in W$ and $y \in W^\perp$, then $x$ is the vector of $W$ that is closest to $v$.

11. (T/F) (8 points) If $v_1$, $v_2$, and $v_3$ are vectors in $\mathbb{R}^4$, and $W$ is the set of all linear combinations of $v_1, v_2, v_3$, then $W$ must be a subspace of $\mathbb{R}^4$.

12. (T/F) (8 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation, let $b$ be a vector in $\mathbb{R}^2$, and let $W$ be the set of all $x \in \mathbb{R}^3$ such that $T(x) = b$, i.e., $W = \{x \in \mathbb{R}^3 \mid T(x) = b\}$. Then $W$ must be a subspace of $\mathbb{R}^3$.

13. (T/F) (8 points) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and let $B$ be an invertible $2 \times 2$ matrix. Then it must be true that $AB = BA$.

14. (T/F) (8 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be a linear transformation such that $T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ and $T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix}$. It is possible that $T(x) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ for some $x \in \mathbb{R}^2$.

15. Let $A$ be a $5 \times 7$ matrix, and let $b$ be a vector in $\mathbb{R}^5$.

(a) (8 points) Is it possible that rank $A = 6$? Give an example of such an $A$, or explain why no such $A$ can exist.

(b) (8 points) Is it possible that the equation $Ax = b$ has exactly one (i.e., at least one, and not more than one) solution $x \in \mathbb{R}^7$? Give an example of such an $A$, or explain why no such $A$ can exist.

16. Suppose that $\{u_1, u_2, u_3\}$ spans a subspace $V$ of $\mathbb{R}^4$.

(a) (9 points) Let $y$ be a vector in $V$. Must it be true that $\{u_1, u_2, u_3, y\}$ spans $V$? Either:
   - Explain why $\{u_1, u_2, u_3, y\}$ must span $V$; or
   - Give a specific example of $\{u_1, u_2, u_3, y\}$ satisfying the above conditions, and explain how you know that $\{u_1, u_2, u_3, y\}$ does not span $V$ in your example.

(b) (9 points) Must it be true that $\{u_1, u_2\}$ spans $V$? Either:
   - Explain why $\{u_1, u_2\}$ must span $V$; or
   - Give a specific example of $\{u_1, u_2, u_3\}$ satisfying the above conditions, and explain how you know that $\{u_1, u_2\}$ does not span $V$ in your example.