Review of span and linear independence
Linear algebra (Math 129A)

Let \( \{ \mathbf{u}_1, \ldots, \mathbf{u}_k \} \) be vectors in \( \mathbb{R}^n \). The fundamental definitions are:

**Definition.** The span of \( \{ \mathbf{u}_1, \ldots, \mathbf{u}_k \} \) is the set of all linear combinations of \( \{ \mathbf{u}_1, \ldots, \mathbf{u}_k \} \). In other words, the span of \( \{ \mathbf{u}_1, \ldots, \mathbf{u}_k \} \) is

\[
\text{Span} \{ \mathbf{u}_1, \ldots, \mathbf{u}_k \} = \{ a_1 \mathbf{u}_1 + \cdots + a_k \mathbf{u}_k \mid a_i \in \mathbb{R} \}.
\]

**Definition.** If, for some \( c_1, \ldots, c_k \in \mathbb{R} \) with not all \( c_i \neq 0 \), we have

\[
c_1 \mathbf{u}_1 + \cdots + c_k \mathbf{u}_k = \mathbf{0},
\]

then we say that \( \{ \mathbf{u}_1, \ldots, \mathbf{u}_k \} \) is **linearly dependent**. If the only solution to (*) is \( c_1 = \cdots = c_k = 0 \), then we say that \( \{ \mathbf{u}_1, \ldots, \mathbf{u}_k \} \) is **linearly independent**.

**Which sets of vectors span \( \mathbb{R}^n \)?** Let \( \{ \mathbf{u}_1, \ldots, \mathbf{u}_k \} \) be vectors in \( \mathbb{R}^n \), and let \( A \) be the \( n \times k \) matrix \( [\mathbf{u}_1 \cdots \mathbf{u}_k] \), i.e., the matrix whose columns are \( \mathbf{u}_1, \ldots, \mathbf{u}_k \). Among other things, the following theorems (Thms. 1.5 and 1.7, respectively) give tests for determining if \( \{ \mathbf{u}_1, \ldots, \mathbf{u}_k \} \) spans \( \mathbb{R}^n \) and determining if \( \{ \mathbf{u}_1, \ldots, \mathbf{u}_k \} \) is linearly independent. (Actually, these tests are really just a single test: finding the rank of \( A \).

**Theorem (Fat Matrix Theorem).** For an \( n \times k \) matrix \( A \), the following are equivalent:

1. The columns of \( A \) span \( \mathbb{R}^n \).
2. For every \( \mathbf{b} \in \mathbb{R}^n \), the equation \( A\mathbf{x} = \mathbf{b} \) has either one solution or infinitely many solutions.
3. \( \text{rank}(A) = n \).
4. \( \text{RREF}(A) \) has no zero rows.

We call this the Fat Matrix Theorem because for the conditions to be true, we must have \( k \geq n \) (i.e., the matrix \( A \) must be “fat”).

**Theorem ( Tall Matrix Theorem).** For an \( n \times k \) matrix \( A \), the following are equivalent:

1. The columns of \( A \) are linearly independent.
2. The only solution to \( A\mathbf{x} = \mathbf{0} \) is \( \mathbf{x} = \mathbf{0} \).
3. For every \( \mathbf{b} \in \mathbb{R}^n \), the equation \( A\mathbf{x} = \mathbf{b} \) has either no solutions or one solution.
4. \( \text{rank}(A) = k \).
5. Every column of \( \text{RREF}(A) \) is a pivot column.

We call this the Tall Matrix Theorem because for the conditions to be true, we must have \( n \geq k \) (i.e., the matrix \( A \) must be “tall”).

**Enlarging or shrinking spanning sets.** Here, we start to see how the ideas of span and linear independence complement each other.

Let \( \{ \mathbf{u}_1, \ldots, \mathbf{u}_k \} \) be vectors in \( \mathbb{R}^n \). In Thm. 1.8, we see that:

**Theorem.** The vectors \( \mathbf{u}_1, \ldots, \mathbf{u}_k \) are linearly dependent precisely if one of the following conditions is true:

1. Either \( \mathbf{u}_1 = \mathbf{0} \), or
2. Some \( \mathbf{u}_i \) (\( 2 \leq i \leq k \)) is a linear combination of the previous vectors.

Combining this with part (c) of Thm. 1.6, we see that:
**Theorem.** Let $S$ be a finite set of vectors in $\mathbb{R}^n$, and let $V = \text{Span } S$. Then $V$ can be spanned by a smaller subset of $S$ if and only if $S$ is linearly dependent.

**Proof.** If $S$ is linearly dependent, then either some vector in $S$ is equal to 0 or at least one vector $z \in S$ is a linear combination of the others. By Thm. 1.6(c), we can remove $z$ from $S$ and obtain a smaller set of vectors with the same span.

Conversely, suppose we can remove a vector $z$ from $S$ and obtain a smaller set of vectors with the same span. In that case, by Thm. 1.6, $z$ is a linear combination of the other vectors in $S$, so by Thm. 1.8, $S$ is linearly dependent. □

**The Span-Independence Theorem.** Another key relationship between spanning and linear independence is Thm. 1.9, whose importance will become clearer later.

**Theorem (Span-Independence Theorem).** Let $\{u_1, \ldots, u_k\}$ be vectors in $\mathbb{R}^n$, and let $V = \text{Span } \{u_1, \ldots, u_k\}$. Every subset of $V$ containing more than $k$ vectors is linearly dependent.

In other words, put in terms of linear independence:

**Theorem.** Let $V$ be a subset of $\mathbb{R}^n$. Any set $\{v_1, \ldots, v_m\}$ that spans $V$ is at least as large as any linearly independent subset $\{w_1, \ldots, w_k\}$ of $V$.

The point is, we do not assume that the $v$'s have any direct relation to the $w$'s (e.g., $v_1$ need not be $w_1$, etc.), but we still know that there have to be more $v$'s than $w$'s.