Supplemental notes on chapter 3  
Math 129B

The foundations of linear algebra. As advertised:

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<th>dimension</th>
<th>basis</th>
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<td>linear combinations</td>
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axioms of a vector space

**Dimension 13.** Suppose $V$ is a vector space with $\dim V = 13$. We then know the following things about $V$ (reasons in parentheses).

- $V$ has some basis $\{v_1, \ldots, v_{13}\}$. (Definition of dimension)
- Since $\dim V > 0$, $V$ has infinitely many different bases. (Discussed in class)
- Any basis for $V$ has 13 vectors in it. (Comparison Thm)
- Any linearly independent set in $V$ can contain at most 13 vectors. (Comparison Thm)
- Any set that spans $V$ must contain at least 13 vectors. (Comparison Thm)
- Given any linearly independent set in $V$, we can add vectors (possibly zero of them) to obtain a basis for $V$. (Expansion Thm)
- Given any spanning set for $V$, we can remove vectors (possibly zero of them) to obtain a basis for $V$. (Contraction Thm)
- Any subspace $W$ of $V$ is finite-dimensional, with $\dim W \leq 13$. In particular, if $\dim W = 13$, then $W = V$. (Subspace Size Thm)
- If a set of 13 vectors spans $V$, then that set must also be linearly independent; similarly, if a set of 13 vectors in $V$ is linearly independent, that set must also span $V$. (Two Out of Three Thm)

**The must/may exercise.** As another review of the meaning of dimension, here are 12 statements. Determine which statements are true and which are false, and give an example or justification for each answer.

Let $W$ be a vector space such that $\dim W = 5$, and suppose that $u_1, u_2, u_3, u_4, u_5, u_6$ are all vectors in $W$.

1. It is possible that $\{u_1, u_2, u_3, u_4\}$ spans $W$.
2. It is possible that $\{u_1, u_2, u_3, u_4, u_5\}$ spans $W$.
3. It is possible that $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ spans $W$.  

4. The set \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} must span \(W\).

5. The set \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\} must span \(W\).

6. The set \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6\} must span \(W\).

7. It is possible that \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} is linearly independent.

8. It is possible that \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\} is linearly independent.

9. It is possible that \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6\} is linearly independent.

10. The set \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} must be linearly independent.

11. The set \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\} must be linearly independent.

12. The set \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6\} must be linearly independent.