The Whatever Theorem. This says:

The Whatever Theorem. Let $V$ and $W$ be vector spaces, let $\{v_1, \ldots, v_n\}$ be a basis for $V$, and let $w_1, \ldots, w_n$ be vectors in $W$ (possibly equal to each other or $0$). Then there exists a unique linear function $T : V \to W$ such that $T(v_i) = w_i$ for $1 \leq i \leq n$.

The main ideas of the Whatever Theorem are: (1) You can make a linear function do Whatever you want to a basis, and (2) This is essentially the only way to make up a linear function/write down a formula for a linear function.

The SPAM and One-to-one Lemmas. These are somewhat complementary tools for proving facts about a linear function $T$. The SPAM Lemma can be used to prove $T$ is onto, or other facts about the image of $T$, by finding a SPAnning set for the iMage of $T$. The One-to-one Lemma deals with the kernel of $T$.

The SPAM Lemma. If $T : V \to W$ is linear and $\{v_1, \ldots, v_n\}$ spans $V$, then $\{T(v_1), \ldots, T(v_n)\}$ spans $\text{im} \ T$.

The One-to-one Lemma. If $T : V \to W$ is linear, then the following are equivalent:
1. $T$ is one-to-one.
2. $\ker T = \{0\}$.
3. nullity $T = 0$.

The matrix of a linear function. The matrix of the linear function $T$ relative to the bases $B$ (domain) and $B'$ (range) is denoted by $[T]_{B,B'}$. Let $B = \{u_1, \ldots, u_k\}$. Then by definition, we have

$$[T]_{B,B'} = \begin{bmatrix} [T(u_1)]_{B'} & \cdots & [T(u_k)]_{B'} \end{bmatrix}.$$ 

Slogan: “The columns tell you where your basis goes.”

The key property of $A = [T]_{B,B'}$ is that

$$A(B\text{-coordinates of } v) = B'\text{-coordinates of } T(v).$$

Diagram:

\[ \begin{array}{ccc}
V & \xrightarrow{T} & W \\
\downarrow{C_B} & & \downarrow{C_B'} \\
\mathbb{R}^k & \xrightarrow{\mu_A} & \mathbb{R}^n \\
\end{array} \]

Change of basis. Suppose $T : V \to V$ is linear, and that we know the matrix $A = [T]_{B,B}$ of $T$ relative to an old basis $B$. Suppose we want to find the matrix of $T$ relative to some new basis $B'$, i.e., suppose we want to find $A' = [T]_{B',B'}$. First a diagram of what’s going on:

\[ \begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{\mu_A} & \mathbb{R}^n \\
\downarrow{C_{B'}} & & \downarrow{C_{B'}} \\
V & \xrightarrow{T} & W \\
\downarrow{C_B} & & \downarrow{C_B} \\
\mathbb{R}^n & \xrightarrow{\mu_A} & \mathbb{R}^n \\
\end{array} \quad \begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{\mu_A} & \mathbb{R}^n \\
\downarrow{C_{B'}} & & \downarrow{C_{B'}} \\
\text{New basis} & & \text{New basis} \\
\downarrow{\mu_P} & & \downarrow{\mu_P} \\
\mathbb{R}^n & \xrightarrow{A'} & \mathbb{R}^n \\
\end{array} \quad \begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{\mu_A} & \mathbb{R}^n \\
\downarrow{C_{B'}} & & \downarrow{C_{B'}} \\
\text{Old basis} & & \text{Old basis} \\
\downarrow{\mu_P} & & \downarrow{\mu_P} \\
\mathbb{R}^n & \xrightarrow{A'} & \mathbb{R}^n \\
\end{array} \quad \begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{\mu_A} & \mathbb{R}^n \\
\downarrow{C_{B'}} & & \downarrow{C_{B'}} \\
\text{New basis} & & \text{New basis} \\
\downarrow{\mu_P} & & \downarrow{\mu_P} \\
\mathbb{R}^n & \xrightarrow{A'} & \mathbb{R}^n \\
\end{array} \quad \begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{\mu_A} & \mathbb{R}^n \\
\downarrow{C_{B'}} & & \downarrow{C_{B'}} \\
\text{Old basis} & & \text{Old basis} \\
\downarrow{\mu_P} & & \downarrow{\mu_P} \\
\mathbb{R}^n & \xrightarrow{A'} & \mathbb{R}^n \\
\end{array} \]

Thus $A' = P^{-1}AP$. 

Slogan: “The columns tell you where your basis goes.”

The key property of $A = [T]_{B,B'}$ is that

$$A(B\text{-coordinates of } v) = B'\text{-coordinates of } T(v).$$
**Definition.** Let $B = \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \}$ be the old basis, and let $B' = \{ \mathbf{v}'_1, \ldots, \mathbf{v}'_n \}$ be the new basis. The **change-of-basis matrix from the basis $B'$ to the basis $B$** is the matrix $P$ whose $i$th column is $[\mathbf{v}'_i]_B$:

$$P = \begin{bmatrix} [\mathbf{v}'_1]_B & \cdots & [\mathbf{v}'_n]_B \end{bmatrix}.$$  

The key property of $P$ is that

$$P(\mathbf{v}') = \mathbf{v}.$$  

Then the formula in the change-of-basis theorem is:

$$[T]_{B',B} = A' = P^{-1}AP.$$  

**The most important case:** Suppose $V = \mathbb{R}^n$, $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, and the old basis is the standard basis $S = \{ \mathbf{e}_1, \ldots, \mathbf{e}_n \}$. Mercifully, in that case, $[T]_{S,S} = A$.

If we want to find the matrix of $T$ relative to a new basis $B = \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \}$, then $P$ changes $B$-coordinates to $S$-coordinates (mnemonic: “change of basis is just a bunch of B.S.”), and the columns of $P$ are just the vectors in $B$. 
