Problems to be done, but not turned in: (6.2) 1, 5, 10, 15, 20; (6.6) 1, 5, 7.

Problems to be turned in:

1. (6.2) 12.

2. (6.2) 13.

3. Recall that $\mathbb{P}_4$ is the vector space of all polynomials of degree $\leq 4$; in particular, an arbitrary element of $\mathbb{P}_4$ has the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$ ($a_i \in \mathbb{R}$). Let $T: \mathbb{P}_4 \to \mathbb{P}_4$ be defined by the formula
   \[ T(p(x)) = p(x) - p(2) \]
   for all $p(x) \in \mathbb{P}_4$.
   (a) Prove that $T$ is linear.
   (b) Find a basis for $\ker T$, a basis for $\text{im} T$, the rank of $T$, and the nullity of $T$.

4. Let $T : V \to W$ be linear. For $b \in W$, let $S_b$ be the solution set to the equation $T(x) = b$ (i.e., let $S_b = \{ x \in V \mid T(x) = b \}$). Prove that if $x_0 \in V$ is one solution to the equation $T(x) = b$, then
   \[ S_b = x_0 + \ker T = \{ x_0 + v \mid v \in \ker T \}. \]
   (Make sure you do the set containment in both directions.)

5. Recall that $\mathbb{P}$ is the vector space of all polynomials (of any degree). Define linear maps $D : \mathbb{P} \to \mathbb{P}$ and $I : \mathbb{P} \to \mathbb{P}$ by the formulas
   \[ D(p(x)) = p'(x), \quad I(p(x)) = \int_0^x p(t) \, dt. \]
   In other words, $D$ is differentiation, and $I$ is indefinite integration, choosing the constant $C = 0$.
   (a) Give as precise a description as possible of exactly which polynomials are in $\ker D$ and $\text{im} D$. (I.e., your description should let a reader know which polynomials are in $\ker D$ and $\text{im} D$ without requiring the reader to do any computation.) Is $D$ one-to-one? Is $D$ onto?
   (b) Give as precise a description as possible of exactly which polynomials are in $\ker I$ and $\text{im} I$. Is $I$ one-to-one? Is $I$ onto?
   (c) Is $D \circ I = \text{id}_\mathbb{P}$? Is $I \circ D = \text{id}_\mathbb{P}$? Are $D$ and $I$ inverses?

(cont. on other side)
6. Let $V$ be a finite-dimensional vector space, and let $T : V \to V$ be linear. Prove that exactly one of the following is true:

- The equation $T(x) = b$ has a solution $x \in V$ for all $b \in V$.
- $\text{nullity}(T) > 0$.

(Aside: This theorem is known as the Fredholm Alternative.)