More about convergence and divergence tests for series
Math 131A

In these notes, we give alternate versions of some of the most useful convergence/divergence tests. We assume the comparison test as background.

**Theorem 1** (Limit comparison test). Let $\sum a_n$ and $\sum b_n$ be series, and suppose that there exist constants $K \in \mathbb{N}$ and $L, M \in \mathbb{R}$ such that for $n \geq K$, we have that $a_n, b_n > 0$ and

$$0 < L \leq \frac{a_n}{b_n} \leq M.$$  \hfill (1)

Then $\sum a_n$ converges if and only if $\sum b_n$ converges.

**Proof.** Exercise. (Apply the ordinary comparison test.) $\square$

**Corollary 2.** If there exists some $K_1 \in \mathbb{N}$ such that $a_n, b_n > 0$ for $n \geq K_1$, and also $\lim a_n / b_n = C$, where $0 < C < +\infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

**Proof.** Exercise. (Apply Theorem 1; see below for an example of how to show that the required hypothesis holds.) $\square$

We also record the following simplified versions of the ratio and root tests, along with their correspondingly somewhat simpler proofs.

**Theorem 3** (Simplified root test). Let $\sum a_n$ be a series, and suppose

$$\lim |a_n|^{1/n} = L.$$  

1. If $0 \leq L < 1$, then $\sum a_n$ converges absolutely.

2. If $L > 1$ (including $L = +\infty$), then $\sum a_n$ diverges.

**Proof.** We first observe that by the definition of limit and the Archimedean Property, in all cases, for any $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for $n \geq K$, we have

$$|a_n|^{1/n} - L < \epsilon,$$  \hfill (2)

or in other words,

$$L - \epsilon < |a_n|^{1/n} < L + \epsilon.$$  \hfill (3)

In the case $L < 1$, taking $\epsilon = \frac{1-L}{2} > 0$ in (3) and letting $r = \frac{L+1}{2} < 1$, we see that there exists $K \in \mathbb{N}$ such that for $n \geq K$, we have

$$|a_n|^{1/n} < L + \epsilon = r < 1.$$  \hfill (4)

It follows that $|a_n| < r^n$ for $n \geq K$, so $\sum |a_n|$ converges by comparison with the convergent geometric series $\sum r^n$. 

1
In the case $L > 1$, taking $\epsilon = \frac{L - 1}{2} > 0$ in (3) and letting $r = \frac{L + 1}{2} > 1$, we see that there exists $K \in \mathbb{N}$ such that for $n \geq K$, we have

$$1 < r = L - \epsilon < |a_n|^{1/n}. \quad (5)$$

It follows that $|a_n| > r^n$, and therefore, that $\lim |a_n| = +\infty$. Now, if $\lim a_n = 0$, it would follow that $\lim |a_n| = 0$; contradiction. Therefore, $\lim a_n \neq 0$, and $\sum a_n$ diverges by the $n$th term test.

**Theorem 4** (Simplified ratio test). Let $\sum a_n$ be a series such that $a_n \neq 0$ for all $n$, and suppose

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = L.$$

1. If $L < 1$, then $\sum a_n$ converges absolutely.

2. If $L > 1$ (including $L = +\infty$), then $\sum a_n$ diverges.

**Proof.** By the argument in the proof of Theorem 3, we see that for any $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for $n \geq K$,

$$L - \epsilon < \left| \frac{a_{n+1}}{a_n} \right| < L + \epsilon. \quad (6)$$

In the case $L < 1$, again following the proof of Theorem 3, we see that for $r = \frac{L + 1}{2} < 1$, there exists $K \in \mathbb{N}$ such that for $n \geq K$, we have

$$\left| \frac{a_{n+1}}{a_n} \right| < r < 1, \quad (7)$$

or in other words, $|a_{n+1}| < r |a_n|$. An easy induction then shows that for $n \geq K$,

$$|a_n| \leq |a_K| r^{n-K}. \quad (8)$$

Therefore, making the change of variables $m = n - K$, since the geometric series

$$\sum_{n=K}^{\infty} |a_K| r^{n-K} = \sum_{m=0}^{\infty} |a_K| r^m \quad (9)$$

converges, by comparison, so does $\sum a_n$.

In the case $L > 1$, we see by Exercise 9.12 that $\lim |a_n| = +\infty$. By the argument used in the $L > 1$ case of the root theorem, it must be the case that $\lim a_n \neq 0$, so $\sum a_n$ diverges by the $n$th term test.

**Example 5.** Problem: Let $a_n = \frac{7n^{25} + 11n^{12}}{3^n - 5n^{16}}$. Does $\sum a_n$ converge or diverge?

Before we start the problem, we observe that by Asymptotics, $7n^{25} \gg 11n^{12}$ and $3^n \gg 5n^{16}$, so we should suspect (without proof as of yet) that the problem will boil down to the convergence or divergence of $\sum \frac{7n^{25}}{3^n}$. So, letting $b_n = \frac{7n^{25}}{3^n}$ and forgetting about the original problem for the moment, here are two ways to approach $\sum b_n$. 2
1. Applying the Ratio Test, we see that

\[
\frac{b_{n+1}}{b_n} = \frac{\frac{7(n+1)^{25}}{3^{n+1}}}{\frac{7n^{25}}{3^n}} = \left( \frac{3^n}{3^{n+1}} \right) \left( \frac{7(n+1)^{25}}{7n^{25}} \right) = \left( \frac{1}{3} \right) \left( 1 + \frac{1}{n} \right)^{25}.
\]  

(10)

Therefore, by the limit laws and the fact that \( \lim \frac{1}{n} = 0 \),

\[
\lim \frac{b_{n+1}}{b_n} = \lim \frac{1}{3} \left( 1 + \frac{1}{n} \right)^{25} = \frac{1}{3} \left( \lim \left( 1 + \frac{1}{n} \right) \right)^{25} = \frac{1}{3} (1^{25}) = \frac{1}{3} < 1,
\]  

(11)

which means that \( \sum b_n \) converges, by the Ratio Test.

2. Applying the Root Test, we see that

\[
|b_n|^{1/n} = \left( \frac{7n^{25}}{3^n} \right)^{1/n} = \frac{7^{1/n}n^{25/n}}{3} = \frac{7^{1/n}(n^{1/n})^{25}}{3}.
\]  

(12)

Therefore, by the limit laws and the fact that \( \lim n^{1/n} = \lim a^{1/n} = 1 \),

\[
\lim |b_n|^{1/n} = \lim \frac{7^{1/n}(n^{1/n})^{25}}{3} = \frac{(\lim 7^{1/n})(\lim n^{1/n})^{25}}{3} = \frac{1}{3} < 1,
\]  

(13)

which means that \( \sum b_n \) converges, by the Root Test.

Returning to the original problem, we hope that we can now compare \( \sum a_n \) to \( \sum b_n \) via the Limit Comparison Test. So first,

\[
\frac{a_n}{b_n} = \left( \frac{7n^{25} + 11n^{12}}{3^n - 5n^{16}} \right) \div \left( \frac{7n^{25}}{3^n} \right) = \left( \frac{7n^{25} + 11n^{12}}{3^n - 5n^{16}} \right) \left( \frac{3^n}{7n^{25}} \right) = \frac{7n^{25}(3^n) + 11n^{12}(3^n)}{7n^{25}3^n - 35n^{41}} = \frac{1 + \left( \frac{11}{7n^{14}} \right)}{1 - \left( \frac{5n^{16}}{3^n} \right)}.
\]  

(14)
where in the last step, we divide top and bottom by $7n^{25}3^n$. Therefore, by Asymptotics,

$$\lim \frac{a_n}{b_n} = \lim \frac{1 + \left(\frac{11}{7n^{15}}\right)}{1 - \left(\frac{5n^{16}}{3^n}\right)}$$

$$= 1 + \lim \frac{\frac{11}{7n^{15}}}{1 - \lim \frac{5n^{16}}{3^n}}$$

$$= 1 + 0 = 1. \quad (15)$$

Since $0 < 1 < +\infty$, by the Limit Comparison Test, $\sum a_n$ converges if and only if $\sum b_n$ converges. However, since we already showed that $\sum b_n$ converges, $\sum a_n$ converges as well.