Suppose we want to do an induction proof of something like the following:

- Consider the function $f(b, n)$ defined by the following recursive algorithm:
  - If $n = 0$, output the value $f(b, n) = 1$.
  - If $n$ is odd, output the value $f(b, n) = b \ast f(b^2, \lfloor n/2 \rfloor)$, where $\ast$ is multiplication and $\lfloor x \rfloor$ is the greatest integer $\leq x$.
  - Otherwise, output the value $f(b, n) = f(b^2, n/2)$.

Prove that if $n$ is an integer and $n \geq 0$, then $f(b, n) = b^n$.

To outline a proof of a theorem $\text{Thm}(n)$ by (standard) induction on $n$, write the following:

- **Base case:** Verify that $\text{Thm}(n)$ works for the smallest case of $n$.

- **Induction:** We would like to prove that $\text{Thm}(k)$ implies $\text{Thm}(k+1)$. So set up the following Assume/Conclude structure:
  - Assume: (Induction hypothesis) $\text{Thm}(k)$ works (write out what this means).
  - Leave a hopeful blank space in the middle.
  - Conclude: (Induction step) $\text{Thm}(k+1)$ works (write out what this means).

Then to finish the proof, fill in the blank space with an explanation of why assuming $\text{Thm}(k)$ leads logically to conclude $\text{Thm}(k+1)$.

Note that for strong induction, the induction hypothesis becomes:

**Assume:** (Induction hypothesis) For (base case) $\leq n \leq k$, $\text{Thm}(n)$ works (write out what this means).

For example, if the base case is $n = 1$, the strong induction hypothesis becomes:

**Assume:** $\text{Thm}(1)$ works, $\text{Thm}(2)$ works, $\ldots$, $\text{Thm}(k-1)$ works, and $\text{Thm}(k)$ works.

*Examples.* An induction outline is relatively straightforward for more mechanical proofs (e.g., proving equations involving summations), so we give two less mechanical examples.

**Example 1:** Suppose we want to prove:

**Theorem:** For $n \geq 1$, the complete graph $K_n$ has $\frac{n(n-1)}{2}$ edges.

The setup (standard induction) is:

- **Base case:** For $n = 1$, $K_1$ has 0 edges, and $\frac{1(0)}{2} = 0$, which checks.

- **Induction:**
– **Assume:** The complete graph $K_k$ has $\frac{k(k-1)}{2}$ edges.

– Hopeful blank space:

(???)

– **Conclude:** The complete graph $K_{k+1}$ has $\frac{(k+1)k}{2}$ edges.

**Example 2:** Here’s the outline for the $f(b, n)$ problem mentioned above.

- **Base case:** For $n = 0$, $f(b, 0)$ returns $f(b, 0) = 1$, and $b^0 = 1$, so $f$ returns the correct answer.

- **Induction step:**
  
  – **Assume:** For $0 \leq n \leq k$, Thm($n$) works. In other words, for $0 \leq n \leq k$, $f(b, n) = b^n$.

  – Hopeful blank space:

  (???)

  – **Conclude:** Thm($k + 1$) works, i.e., $f(b, k + 1) = b^{k+1}$. 