ON LINEAR AND RESIDUAL PROPERTIES OF GRAPH PRODUCTS

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Abstract. We show that the graph product of subgroups of Coxeter groups is a subgroup of a Coxeter group. As a result, we obtain short proofs that graph groups (right-angled Artin groups) are linear and that the graph product of residually finite groups is residually finite. We also give a new and more geometric proof of the normal form theorem for graph products.

1. Introduction

Graph groups are groups with presentations where the only relators are commutators of the generators. Graph groups were first investigated by Baudisch [1], and much subsequent foundational work was done on them by Droms, B. Servatius, and H. Servatius [3, 4, 5]. Later, the more general construction of graph products (Definition 2.1) was introduced and developed by Green [7]. (Graph products are to free products as graph groups are to free groups.) Graph groups have also been of recent interest because of their geometric properties (Hermiller and Meier [8] and VanWyk [13]) and the cohomological properties of their subgroups (Bestvina and Brady [2]).

In this paper, by embedding graph products in Coxeter groups, we obtain short proofs of several fundamental properties of graph products. Specifically, after listing some preliminary definitions and results in Section 2, we show in Section 3 that the graph product of subgroups of Coxeter groups is a subgroup of a Coxeter group (Theorem 3.2). It follows that many classes of graph products are linear, including graph groups (a result of Humphries [11]); and that the graph product of residually finite groups is residually finite (a result of Green [7]). In Section 4, we also include a new and more geometric proof of Green’s normal form theorem for graph products. Finally, in Section 5, we list some related open problems.

2. Graph products

In this section, we review some basic definitions and results on graph products.

For a simplicial graph $\Gamma$, we let $\Gamma^0$ denote the vertices of $\Gamma$, we let $\Gamma^1$ denote the edges of $\Gamma$, and we let $[v, w]$ denote the edge between the vertices $v$ and $w$.

Definition 2.1. Let $\Gamma$ be a finite simplicial graph, and for each $v \in \Gamma^0$ let $G_v$ be a group called the vertex group of $v$. The graph product $\Gamma G_v$ is defined to be the free product of the $G_v$, subject to the relations

\[ [g_v, g_w] = 1 \quad \text{for all } g_v \in G_v, g_w \in G_w \text{ such that } [v, w] \in \Gamma^1. \]

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In particular, if $G_v \cong \mathbb{Z}$ for all $v \in \Gamma^0$, $\Gamma G_v$ is called a graph group or right-angled Artin group. If $G_v \cong \mathbb{Z}/2$ for all $v \in \Gamma^0$, then $\Gamma G_v$ is a Coxeter group with all edges labeled either 2 or $\infty$. Such groups are known as right-angled Coxeter groups.

**Definition 2.2.** Let $\Gamma$ be a finite simplicial graph, and let $G_v$ and $C_v$ be two sets of vertex groups for $\Gamma$ such that there exists some homomorphism $\varphi_v : G_v \to C_v$ for each $v \in \Gamma^0$. The natural map from $\Gamma G_v$ to $\Gamma C_v$ is the unique homomorphism which restricts to $\varphi_v$ on each of the $G_v$. (The existence of such a map follows easily from the definition of graph product.)

Now, by definition, any element $g$ of a graph product $\Gamma G_v$ can be represented as a product $g_1 g_2 \ldots g_m$, where each $g_i$ is an element of some vertex group $G_v$. Definition 2.3, Definition 2.4, and Theorem 2.5 describe how to do so in the “shortest” possible way.

**Definition 2.3.** If $g$ is an element of a graph product $\Gamma G_v$, then we may represent $g$ by a product $W = g_1 g_2 \ldots g_m$ of elements $g_i$, each of which is an element of some vertex group $G_v$. $W$ is called a word representing $g$, and the $g_i$ are called the syllables of $W$. The number of syllables in $W$ is called the length of $W$.

Note that each of the following “moves” changes a given word $W$ to a word $W'$ which represents the same element of $\Gamma G_v$ as $W$ does, and has length less than or equal to the length of $W$.

1. Remove a syllable $g_i = 1$.
2. Replace consecutive syllables $g_i$ and $g_{i+1}$ in the same vertex group $G_v$ with the single syllable $(g_i g_{i+1})$.
3. For consecutive syllables $g_i \in G_v$ and $g_{i+1} \in G_w$ such that $[v, w] \in \Gamma^1$, exchange $g_i$ and $g_{i+1}$.

**Definition 2.4.** If $g$ is represented by a word $W$ which cannot be changed to a shorter word using any sequence of the above moves, then $W$ is said to be a normal form for $g$.

We give a geometric proof of the following theorem of Green [7] in Section 4. For the moment, we will be content just to quote it.

**Theorem 2.5.** A normal form in a graph product represents the trivial element if and only if it is the empty word.

Finally, we need the following definition:

**Definition 2.6.** Let $\Lambda$ and $\Gamma$ be simplicial graphs, and let $G_v$ (resp. $G_w$) be vertex groups for $\Lambda$ (resp. $\Gamma$). A full inclusion is an inclusion $\rho : \Lambda \to \Gamma$ of simplicial graphs such that for any two vertices $u, v \in \Lambda$, if $[\rho(u), \rho(v)] \in \Gamma^1$ then $[u, v] \in \Lambda^1$. If $\rho : \Lambda \to \Gamma$ is a full inclusion and $G_v \cong G_{\rho(v)}$ for all $v \in \Lambda^0$, then $\Lambda G_v$ is called a full subgroup of $\Gamma G_w$. Note that $\Lambda G_v$ is indeed a subgroup of $\Gamma G_w$, since the homomorphism induced by $\rho$ maps normal forms to normal forms.

3. **Graph products of Coxeter subgroups**

**Definition 3.1.** By a Coxeter subgroup we mean a subgroup of a Coxeter group.

For example, any finite group $G$ is a Coxeter subgroup, since $G$ is a subgroup of some symmetric group; and any (possibly infinite) cyclic group $G$ is a Coxeter subgroup, since $G$ is a subgroup of some (possibly infinite) dihedral group. Note
that since Coxeter groups are linear (subgroups of $GL_n(\mathbb{R})$) and residually finite, so are Coxeter subgroups.

**Theorem 3.2.** The graph product of Coxeter subgroups is a Coxeter subgroup.

**Proof.** Let $\Gamma G_v$ be a graph product such that, for each $v \in \Gamma^0$, $G_v$ is a subgroup of the Coxeter group $C_v$ with reflection generators $\{r_{vi}\}$. Consider the Coxeter group $C$ with reflection generators $\{r_{vi}\}$, where $v$ runs over all $v \in \Gamma^0$, and Coxeter relations

$$
\text{order}(r_{vi}r_{vj}) = \begin{cases} 
\text{order}(r_{vi}r_{vj}) \text{ in } C_v & \text{for } v = w \\
2 & \text{for } v \neq w, [v, w] \in \Gamma^1 \\
\infty & \text{for } v \neq w, [v, w] \notin \Gamma^1
\end{cases}
$$

By the definition of graph product, $C$ is the graph product $\Gamma C_v$. Since the natural map from $\Gamma G_v$ to $\Gamma C_v$ sends normal forms to normal forms, the theorem follows. □

**Remark 3.3.** Note that Droms and Servatius [6] used a similar construction, in the special case of a graph product of infinite cyclic groups, to show that the Cayley graphs of graph groups are isomorphic (as undirected graphs) to the Cayley graphs of right-angled Coxeter groups. However, their graph isomorphism is not equivariant and does not come from a group homomorphism, and is therefore quite different from our group embedding.

**Example 3.4.** Let $\Gamma G_v$ be the graph group shown on the left-hand side of Figure 1; or in other words, let $\Gamma G_v$ be the indicated graph product of the infinite cyclic groups $\langle a_i \rangle$ ($1 \leq i \leq 4$). Let $C$ be the Coxeter group whose Coxeter diagram is shown on the right-hand side of Figure 1, using the convention that all edges are labelled with $\infty$. Finally, since any infinite cyclic group is a subgroup of the Coxeter group $\infty$, let $\varphi$ be the homomorphism from $\Gamma G_v$ to $C$ which embeds each $\langle a_i \rangle$ in the vertical (thick-line) $\infty$ group labelled $\langle a_i \rangle$ on the right-hand side of Figure 1. Following the recipe given by (2), we see that $\varphi$ embeds $\Gamma G_v$ as a subgroup of $C$. (Note that since graph products and Coxeter groups have opposite graph conventions for commuting relations, $\varphi$ sends joined vertices to non-joined $\infty$ groups, and vice versa.)

![Figure 1. Embedding a graph group in a Coxeter group](image)

Theorem 3.2 allows us to conclude that many graph products are Coxeter subgroups, and therefore linear, residually finite, and so on. For instance:

**Corollary 3.5.** The graph product of finite groups and cyclic groups is a Coxeter subgroup, and therefore linear and residually finite. □

In particular, we recover the following result of Humphries [11].

**Corollary 3.6.** Right-angled Artin groups, or graph groups, are linear. □
In fact, we have actually shown that every right-angled Artin group on \( n \) generators is a subgroup of a right-angled Coxeter group on \( 2n \) generators.

We may also use Theorem 3.2 (or actually, Corollary 3.5) to obtain a short proof of the following theorem of Green [7].

**Theorem 3.7.** The graph product of residually finite groups is residually finite.

**Proof.** Let \( \Gamma G_v \) be a graph product and suppose that each \( G_v \) is residually finite. We wish to show that for \( 1 \neq g \in \Gamma G_v \), \( g \) survives in some finite quotient of \( \Gamma G_v \). Suppose \( g \) has some normal form \( g = g_1 g_2 \cdots g_r \). Choose finite quotients \( Q_v \) of each of the \( G_v \) such that all of the \( g_i \) survive in their respective quotients. The natural homomorphism \( \varphi : \Gamma G_v \rightarrow \Gamma Q_v \) sends \( g \) to an element with a non-trivial normal form, which means that \( \varphi(g) \neq 1 \). Then, since \( \Gamma Q_v \) is residually finite (Corollary 3.5), there is some finite quotient of \( \Gamma Q_v \) in which \( \varphi(g) \) survives, and the theorem follows.

Recall that the *profinite topology* on a group \( G \) is the topology whose closed basis consists of cosets of finite index subgroups of \( G \). Note that \( G \) is residually finite if and only if \( \{1_G\} \) is a closed subset, and more generally, a subgroup \( H \) of \( G \) is closed if and only if \( H \) is the intersection of finite index subgroups of \( G \). Finally, note that a homomorphism of groups is a continuous map relative to their profinite topologies. See Higgins [9] for more about the profinite topology.

Now, Green also extended Theorem 3.7 to show:

**Theorem 3.8** (Green). Let \( G \) be a graph product of residually finite groups, and let \( H \) be a full subgroup of \( G \). Then \( H \) is closed in the profinite topology of \( G \).

We now extend Theorem 3.7 further (Theorem 3.10), using the following lemma.

**Lemma 3.9.** Let \( G \) be a residually finite group, and let \( \varphi : G \rightarrow G \) be a retraction map (i.e., \( \varphi^2 = \varphi \)). Then:

1. \( \varphi(G) \) is closed in the profinite topology of \( G \).

2. Any closed subgroup of \( \varphi(G) \) in the profinite topology on \( \varphi(G) \) is also closed as a subgroup of \( G \). In other words, the inclusion map \( \varphi(G) \hookrightarrow G \) is a homeomorphism with respect to the profinite topologies of the two groups.

**Proof.** Since \( H = \varphi(G) \) is a retract of \( G \), if \( N = \ker \varphi \), then \( G = NH \) and \( N \cap H = 1 \). Using the residual finiteness of \( G \), let \( G_i \) be a sequence of finite index normal subgroups of \( G \) whose intersection is 1, and let \( N_i = N \cap G_i \). Then since

\[
[G : N_i H] = [NH : N_i H] = [N : N_i] \leq [G : G_i],
\]

\( N_i H \) is a sequence of finite index subgroups of \( G \). However, since any element of \( G \) is uniquely expressed as a product \( nh \) \((n \in N, h \in H)\), the intersection of the \( N_i H \) is precisely \( H \). Statement (1) follows.

As for (2), let \( K \) denote the subgroup \( \varphi(G) \) equipped with its own profinite topology, and let \( L \) be a closed subgroup of \( K \). Since the homomorphism \( \varphi : G \rightarrow K \) is continuous, \( \varphi^{-1}(L) \) is the preimage of a closed set. Then, since \( L \) is the intersection of the closed subgroups \( K \) and \( \varphi^{-1}(L) \) of \( G \), \( L \) must also be closed in \( G \). The lemma follows.

**Theorem 3.10** (Green). Let \( \Lambda \) be a full subgraph of \( \Gamma \) (Definition 2.6), and for \( v \in \Gamma^0 \), let \( G_v \) be residually finite. Then the inclusion of \( \Lambda G_v \) as a full subgroup of \( \Gamma G_v \) is a homeomorphism with respect to the profinite topologies of the two groups.
Proof. For each $v \in \Gamma^0$, let $\psi_v : G_v \to G_v$ be the identity if $v \in \Lambda^0$, and trivial otherwise. Then the resulting natural map $\varphi$ is a retraction from $\Gamma G_v$ onto $\Lambda G_v$, and the theorem follows from Lemma 3.9. \hfill \Box

Remark 3.11. In a future paper [10], we will provide a more extensive answer to the question of which subgroups of a graph group $\Gamma G_v$ are closed. Specifically, we hope to show that any subgroup of $\Gamma G_v$ which has finitely generated intersection with every conjugate of every full subgroup of $\Gamma G_v$ is closed in $\Gamma G_v$.

4. PROOF OF THE NORMAL FORM THEOREM

In this section, we give a proof of Theorem 2.5 based on the geometry of van Kampen diagrams. Throughout this section, we fix a graph product $\Gamma G_v$, and we use the presentation of $\Gamma G_v$ obtained by combining the “multiplication table” presentations of the $G_v$ and the commutators in (1), Definition 2.1. Relators of the first type we call multiplication relators, and relators of the second type we call graph relators.

Throughout this section, we consider a word $W$ (Definition 2.3) which represents the trivial element of $\Gamma G_v$, and a van Kampen diagram $D$ for $W$. That is, following Lyndon and Schupp [12], we consider a singular disc diagram $D$ (with basepoint $d \in \partial D$), made by “sticking together” relators from the presentation of $\Gamma G_v$, such that $W$ is the label of a closed path around $\partial D$ beginning and ending at $d$. (Note that because of our chosen basepoint $d$, there is a well-defined notion of being “between” two syllables of $\partial D$.)

Definition 4.1. For $v \in \Gamma^0$, we define the diagram $D_v$ to be the disjoint union of all 2-cells of $D$ which correspond to 2-cells coming either from a multiplication relator in $G_v$ or from a graph relator $[g_v, g_w]$ ($g_v \in G_v$), identifying two 2-cells along an edge $e$ if and only if their images in $D$ intersect along $e$.

Definition 4.2. We define a $v$-component of $D$ to be a component of $D_v$. For a $v$-component $C$, we define $\partial_v C$ (the “outer boundary” of $C$) to be the set of edges of $\partial C$ which are mapped to $\partial D$ and which also correspond to elements of $G_v$.

![Figure 2. A $v$-component mapped into $D$](image)

Note that a $v$-component $C$ is not necessarily a subdiagram of $D$, since extra identifications may occur in $D$ along 0-cells of $C$. Figure 2 gives an example of a $v$-component which has such extra identifications when mapped into $D$. (Solid edges correspond to elements of $G_v$, and dashed edges correspond to other elements.)

Note also that for a $v$-component $C$, $\partial_v C$ may be a disconnected, proper subset of $\partial C$. Nevertheless, since the cyclic ordering on $\partial D$ determines a cyclic ordering of the edges of $\partial_v C$, by concatenating the edges of $\partial_v C$, we obtain a closed directed
path $\partial_e C$ (the “closed outer boundary” of $C$). Now, since each of the edges of $\partial_e C$ is labelled by an element of $G_v$, $\partial_e C$ represents a conjugacy class of $G_v$. We can therefore state the following key lemma:

**Lemma 4.3.** If $C$ is a $v$-component, then $\partial_e C$ represents the trivial element of $G_v$.

**Proof.** Let $q$ be the map defined by quotienting each of the 2-cells of $C$ of the form $[g_v, g_w]$ to a 1-cell $g_v$, or in other words, by retracting each graph relator $[g_v, g_w]$ along its two $g_w$ sides. It is easy to see that the resulting quotient $q(C)$ is a diagram made by sticking together multiplication relators from $G_v$. Therefore, it is enough to show that all of the edges in the boundary of $q(C)$ come from edges in $\partial_e C$, for then $\partial(q(C))$ has one component, $q(C)$ is a van Kampen diagram in the presentation of $G_v$, and $\partial(q(C)) = \partial_e C$. (Note that the cyclic ordering of the edges of $\partial_e C$ determined by $\partial D$ is the same as the cyclic ordering of these edges in $\partial(q(C))$.)

![Figure 3. Are there boundary edges on the inside of $D$?](image)

Now, if there is some edge $e$ in the boundary of $q(C)$ such that $q^{-1}(e) \cap \partial_e C = \emptyset$, then $q^{-1}(e)$ must include some edge $e'$ such that the image of $e'$ in $D$ is both on the boundary of the image of $C$ and also on the inside of $D$, as shown by the heavy edges in Figure 3. However, since any edge on the inside of $D$ corresponding to an element of $G_v$ must border a 2-cell coming either from a multiplication relator of $G_v$ or from a graph relator $[g_v, g_w]$ ($g_v \in G_v$) on both sides, no such edge $e'$ exists. The lemma follows. □

**Proof of Theorem 2.5.** Let $W$ be a word which represents the trivial element of $\Gamma G_v$. First, reduce $W$ as much as possible by moves of type 1 and 2 (see the list before Definition 2.4). Then, if $C_v$ is a $v$-component, because $\partial_v C_v = 1$ in $G_v$ (Lemma 4.3) and $W$ cannot be further reduced by moves of type 1 and 2, the image of $\partial_v C_v$ in $D$ is not connected. We may therefore choose some $v \in \Gamma^0$ and some $v$-component $C_v$ with syllables $g_v, g'_v \in \partial_v C_v$ such that $g_v$ and $g'_v$ are innermost, i.e., such that there is no syllable from $\partial_v C_v$ between $g_v$ and $g'_v$, and there is no $w$-component $C_w$ such that $\partial_w C_w$ contains syllables $g_w$ and $g'_w$ which are both between $g_v$ and $g'_v$.

Now let $g_v$ be a syllable between $g_v$ and $g'_v$, and let $C_w$ be the $w$-component containing $g_w$. As before, since $\partial_w C_w = 1$ in $G_w$, $\partial_w C_w$ must have at least two components. Furthermore, only one component of $\partial_w C_w$ can be between $g_v$ and $g'_v$, since $g_v$ and $g'_v$ are innermost. The image of $C_w$ must therefore intersect the image of $C_v$ at a 2-cell (see Figure 4), which implies that $[v, w] \in \Gamma^1$. In other words, for all syllables $g_w$ between $g_v$ and $g'_v$, we see that $g_w$ commutes with $g_v$. Therefore, using moves of type 3, we may change $W$ to a word $W' = \ldots g_v g'_v \ldots$,
and then using a move of type 2, we may make $W'$ shorter. The theorem follows by induction on the length of $W$.

5. PROBLEMS

In closing, we raise two questions.

1. Is the finite graph product of finitely generated linear groups linear? Clearly the direct product of linear groups is linear, and it is also known that the free product of linear groups is linear (Wehrfritz [14]).

2. Are Artin groups linear? Are they residually finite? Note that a special case of the first question is the long-standing open question of whether braid groups are linear. Also, an affirmative answer to either of these questions would produce a solution to the word problem for Artin groups. More speculatively, we ask: are Artin groups Coxeter subgroups?

REFERENCES


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