Complex Number \( \sqrt{-1} = i \in \mathbb{C} \)

**Definition of a complex number.**

If \( a \) and \( b \) are real numbers, the number \( a + bi \) is called a complex number and it is said to be written in standard form. If \( b = 0 \), the number \( a + bi = a \) is a real number. If \( b \neq 0 \), the number \( a + bi \) is called an imaginary number. A number of the form \( bi \), where \( b \neq 0 \), is called a pure imaginary number.

**Equality of a complex number.**

Two complex number \( a + bi \) and \( c + di \) written in standard form are equal to each other, that is, \( a + bi = c + di \)

\[ \iff a = c \text{ and } b = d. \]

i.e.: \( 2 + 3i = 2 + 3i \)

\[ \iff 2 = 2 \text{ and } 3 = 3. \]
Addition and subtraction of complex numbers.

If \(a+bi\) and \(c+di\) are 2 complex numbers written in standard form,

**sum:** \((a+bi) + (c+di) = a+c + (b+d)i\)

Why is that true?

because \(\text{LHS} = (a+bi) + (c+di) = a+bi + c+di\),

\[= a+c + b+i + d\]

\[= a+c + (b+d)i = \text{RHS}.\]

**difference:** \((a+bi) - (c+di) = a+bi - c-di = a-c + bi-di\)

\[= (a-c) + (b-d)i\]

so, \((a+bi) - (c+di) = (a-c) + (b-d)i\).
Recall the definition of an imaginary number
\[ \sqrt{-1} = i \], let's do some easy arithmetic.

1. \( \sqrt{-1} = i \)
2. \( i^2 = (\sqrt{-1})^2 = -1 \)
3. \( i^3 = (\sqrt{-1})^3 = \sqrt{-1} \cdot \sqrt{-1} \cdot \sqrt{-1} = -1 \cdot i = -i \)
4. \( i^4 = i^2 \cdot i^2 = (\sqrt{-1})^2 \cdot (\sqrt{-1})^2 = -1 \cdot -1 = 1 \)
5. \( 1 + 2i + 3 + 4i = 1 + 3 + 2i + 4i = 4 + 6i \)
6. \( 1 - 2i + 5 - 10i = 1 + 5 - 2i - 10i = 6 - 12i \)

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for multiplication of \((a + bi) \cdot (c + di)\)

\[ i^2 = -1 \]

\[ = ac + ad i + bc i + bd i^2 \]

\[ = ac + (ad + bc) i - bd \]

\[ = (ac - bd) + (ad + bc) i \]
Perform the indicated operation.

**(Ex 1)** \((13-2i) + (5+6i)\)
\[= 13 - 2i - 5 + 6i\]
\[= 13 - 5 - 2i + 6i\]
\[= 8 + 4i\]

**(Ex 2)** \((6-5i)(1+i)\)
\[= 6 + 6i - 5i - 5i^2\]
\[= 6 + i - 5(-1)\]
\[= 6 + i + 5\]
\[= 11 + i\]

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Complex conjugate (the opposite of standard form).

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<th>Standard form</th>
<th>Corresponding complex conjugate</th>
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<td>1. (a+bi)</td>
<td>(a-bi)</td>
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<td>2. (a-bi)</td>
<td>(a+bi)</td>
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Write the complex number in standard form and find its complex conjugate.

**(Ex 3)** \(2 + \sqrt{-25}\)
\[= 2 + \sqrt{-1}(5)(5)\]
\[= 2 + 5i\]
\[\Rightarrow 2 - 5i\]
Ex 24 \((\sqrt{2}-4)^2 - 5\)

\[= -4 - 5 \cdot \circled{-9} \quad \text{complex conjugate}\]
\[\implies = -9 \quad \text{also equal to} \quad -9 + 0i \quad \text{and} \quad -9 - 0i \quad = -9\]

Write the quotient in standard form.

Ex 5 \(\frac{(3-2i)(2+5i)}{4+3i}\) = \(\frac{6+15i - 2i - 5i^2}{4+3i}\)

\[= \frac{6+13i+5}{4+3i} = \frac{11+13i}{4+3i} \cdot \frac{4-3i}{4-3i}\]
\[= \frac{44-33i+52i+39}{16+9} = \frac{83+19i}{25}\]
\[= \frac{83}{25} + \frac{19}{25}i\]

Ex 6 solve the quadratic equation
\[x^2 + 6x + 10 = 0\]
\[x = \frac{-6 \pm \sqrt{36 - 40}}{2} = \frac{-6 \pm \sqrt{-4}}{2}\]
\[= \frac{-6 \pm 2i}{2} \implies -3 \pm i\]
How to graph a complex number.

1. $3 + 2i$
2. $-1 - 3i$
3. $-2 + i$
Solve the following equations for real numbers $x$ and $y$.

1) $(3+4i)^2 - 2(x-iy) = x+iy$.

$(3+4i)(3+4i) - 2x + 2iy = x+iy$

$9+12i+12i+16i^2 = x+2x+iy-2iy$

$9+24i-16 = 3x-iy$

$-7+24i = 3x-iy$

So $-7=3x$ and $24i=-iy$.

$x=-\frac{7}{3}$

$\frac{24}{y}=-y$

$y=-24$

2) $\left(\frac{1+i}{1-i}\right)^2 + \frac{1}{x+iy} = 1+i$.

$\frac{(1+i)(1+i)}{(1-i)(1+i)} - 1-i = \frac{-1}{x+iy}$

$\frac{1+2i+i^2}{1-2i+i^2} - 1-i = \frac{-1}{x+iy}$

$\frac{1+2i-1}{1-2i-1} - 1-i = \frac{-1}{x+iy}$
\[-1 - 1 - i = \frac{-1}{x + iy}\]
\[-2 - i = \frac{-1}{x + iy}\]

Note that \( A = \frac{B}{C} \iff C = \frac{B}{A}\)

Thus, \( A = -2 - i, \ C = x + iy, \ B = -1\)

so, \( x + iy = \frac{-1}{-2 - i} \cdot \frac{-2 + i}{-2 + i}\) (complex conjugate)

\( x + iy = \frac{-2 + i}{4 - i^2}\)

\( x + iy = \frac{-2 + i}{5}\)

\( x + iy = \frac{-2}{5} + \frac{1}{5}i\)

\( x = \frac{-2}{5}, \ y = \frac{1}{5}\)

\( \text{Done 2} \)
Argand Diagrams.

Recall that a complex number \( z \) can be represented by \( z = a + bi \), where \( a, b \in \mathbb{R} \), \( i \in \mathbb{C} \).

In a geometric representation, we let \( z = x + iy \), where \( x, y \in \mathbb{R} \) and \( i \in \mathbb{C} \).

There are 2 geometric representations of the complex number \( z = x + iy \):

1. The point \( P(x, y) \) in the xy-plane.
2. The vector \( \overrightarrow{OP} \) from the origin to \( P \).

Vector is a quantity represented as an arrow with direction and magnitude.

Let's draw the Argand diagram.
Let's recall the property of $\cos \theta$ and $\sin \theta$

\[ \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}, \quad \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \]

\[ \cos \theta = \frac{x}{r} \quad \sin \theta = \frac{y}{r} \]

$\iff x = r\cos \theta$  $\iff y = r\sin \theta$

Now recall $z = x + iy$, plug in $x$ and $y$

We get $z = r\cos \theta + ir\sin \theta$

$z = r(\cos \theta + i\sin \theta)$, this is another way to represent a complex number $z$ in the Argand Diagram.

We define the absolute value of a complex number $x + yi$ to be the length $r$ of a vector $\overrightarrow{OP}$ from the origin to $P(x, y)$.

\[ |x + yi| = \sqrt{x^2 + y^2} \]

why is this true?
Reason 1: \( z = x + yi \) represents the length of \( r \) in the Argand diagram, and length cannot be negative; therefore, \( |z| = |x + yi| \) guarantees the length to be positive.

Also, what is another way to find the length of \( r \) in the diagram?

\[ x^2 + y^2 = r^2 \]

Answer: Pythagorean theorem

\[ r = \pm \sqrt{x^2 + y^2} \]

Since length cannot be negative,

So \( r = \sqrt{x^2 + y^2} \)

Connect reason 1 and reason 2, we get \( |x + yi| = \sqrt{x^2 + y^2} \)

The polar angle \( \theta \) has a new name called argument of \( z \) and is written as \( \theta = \arg z \).
\[
\sqrt{5} = \sqrt{1^2 + 2^2} = \sqrt{5}
\]

So, \(|z|^2 = |1 + 2i|^2 = \sqrt{1^2 + 2^2} = \sqrt{5}\)

Then \(|z| = |1 + 2i| = \sqrt{5}\)

For \(|z|^2 = 5\), Recall \(|z| = \sqrt{5}\) so \(z^2 = \sqrt{5}\).

Example: \(z = 1 + 2i\), then \(\bar{z} = 1 - 2i\)

\[|z|^2 = |\bar{z}|^2\]

Its corresponding complex conjugate. Connection between a complex number \(z\) and
Euler's Formula: \( e^{i\theta} = \cos \theta + i\sin \theta \)

Recall \( z = x + iy = r(\cos \theta + i\sin \theta) \).

So, \( z = re^{i\theta} \) by Euler's Formula.