Anosov Flows of Codimension One

by

Slobodan Simić

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Professor Charles C. Pugh, Chair
Professor Morris W. Hirsch
Professor Charles Desoer

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Abstract

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The main goal of this dissertation is to show the existence of global cross sections for certain classes of Anosov flows. Let $\Phi$ be a $C^2$ codimension one Anosov flow on a compact Riemannian manifold $M$ of dimension greater than three. Verjovsky conjectured that $\Phi$ admits a global cross section and we affirm this conjecture in the following three cases: 1) if the sum, $E^{su}$, of the strong stable and strong unstable bundle of $\Phi$ is Lipschitz; 2) if $E^{su}$ is $\theta$-Hölder continuous for all $\theta < 1$ and $\Phi$ preserves volume; 3) if the center stable distribution of $\Phi$ is of class $C^{1+\theta}$ for all $\theta < 1$ and $\Phi$ preserves volume. We note that 1) and 3) generalize the results of Ghys from [Gh3]. For showing 1), we needed to prove a natural generalization of the theorem of Frobenius on integrability of distributions which are only Lipschitz.

We also show how certain transitive Anosov flows (those whose center stable distribution is $C^1$ and transversely orientable) can be “synchronized”, that is, reparametrized so that the strong unstable determinant of the time $t$ map (for all $t$) of the synchronized flow is identically equal to $e^t$. Several applications of this method are given, including vanishing of the Godbillon-Vey class of the center stable foliation of a codimension one Anosov flow (when $\dim M > 3$ and that foliation is $C^{1+\theta}$ for all $\theta < 1$), and a positive answer to a higher dimensional analog to Problem 10.4 posed by Hurder
and Katok in [HK].

We also prove that, under an additional assumption, the lift of the synchronization of $\Phi$ to the universal covering space of $M$ admits a global Lyapunov function which strictly increases along the orbits and is constant on the lifts of the strong stable leaves.
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Dedication

To my parents
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Introduction

In this dissertation we will mainly be concerned with the following conjecture originally posed by Alberto Verjovsky in the early seventies and also stated by Étienne Ghys in [Gh3]:

**Conjecture of Verjovsky**  *Every codimension one Anosov flow on a compact manifold of dimension greater than three admits a global cross section.*

Anosov dynamical systems (originally called the “U-systems”) have been intensively studied since the early sixties and in particular after D.V. Anosov published his influential paper [An] in which he proved most of the fundamental results such as their structural stability, ergodicity, absolute continuity of the strong Anosov foliations, density of periodic orbits in the nonwandering set, etc. Codimension one Anosov systems, i.e. those Anosov systems for which one of the Anosov distributions is of codimension one, have been a particularly popular object of study. Verjovsky’s paper [Ve] is an important contribution, proving, among other things, a remarkable property of codimension one Anosov flows; namely, that they are transitive if the dimension of the underlying manifold is greater than three. (This requirement on the dimension is necessary: Franks and Williams constructed an example of a non-transitive Anosov flow on a 3-manifold.)

Codimension one Anosov diffeomorphisms were successfully classified by Franks [Fra] and Newhouse [Nh]. They showed that every such map is topologically conjugate to a linear hyperbolic automorphism of the torus of the appropriate dimension. (A similar result was later proved by A. Manning, namely: “There are no new Anosov diffeomorphisms on tori.”) In the continuous time case, no examples are known of codimension one Anosov flows in dimensions greater than three other than those obtained by suspending Anosov diffeomorphisms. Again, the dimensional requirement
is necessary because geodesic flows for surfaces of negative curvature are Anosov but have no cross sections. That prompted Verjovsky (and Ghys) to pose the conjecture above, realizing that if true, it would imply complete classification of codimension one Anosov flows from the dynamical viewpoint.

In his thesis [Pl1], J. Plante established several criteria for existence of global cross sections to Anosov flows. Later on, he proved the conjecture of Verjovsky for manifolds with solvable fundamental group. (Similar results were also obtained by Armendariz [Ar].) Further evidence supporting a general feeling that the conjecture is true was given by É. Ghys [Gh3] who proved it with some extra differentiability of the Anosov distributions.

In this paper we intend to generalize certain known results (contained mostly in [Gh3]) and prove several new ones. In Chapter 1, we give the necessary preliminaries and discuss methods for finding cross sections we plan to use. The main results of this chapter are a natural extension of the classical Frobenius theorem on integrability of vector distributions to the “Lipschitz universe” and, as a consequence, a generalization of a result from [Gh3].

In Chapter 2 we show that Verjovsky’s conjecture holds for volume preserving flows for which the sum of the strong distributions is Hölder continuous for all exponents less than one.

Chapter 3 discusses a method of “synchronizing” Anosov flows by which one of the canonical Anosov cocycles (more precisely, the determinant of the flow on its strong unstable bundle) is made “constant”, that is, independent of the space variable. This is used to generalize another result from [Gh3], as well as to show that if the center stable foliation of a codimension one Anosov flow in higher dimensions is \( C^{1+\text{Lip}^{-}} \), then its Godbillon-Vey class vanishes. We also give an answer to a higher dimensional analog to Problem 10.4 from [HK].

In the last chapter, we study the lift of a codimension one Anosov flow to the universal covering space of the underlying manifold and show that if the lift of the center stable foliation admits a transversal intersecting all its leaves, then the lifted flow has a global Lyapunov function which strictly increases along its orbits and is constant on the leaves of the lifted center stable foliation of the synchronized flow.
Chapter 1

Anosov Flows with Lipschitz Distributions

1.1 Definitions and Basic Facts

We first recall the basics.

Definition 1.1 A nonsingular $C^1$ flow $\{f_t\}$ on a compact Riemannian manifold $M$ is called an Anosov flow if there is a continuous $f_t$-invariant splitting of the tangent bundle of $M$,

$$TM = E^c \oplus E^{ss} \oplus E^{uu},$$

such that:

- The bundle $E^c$ is tangent to the flow.

- There exist constants $A > 0$ and $\lambda > 0$ such that for all $v \in E^{ss}$ and $t > 0$,

$$|Tf_t(v)| \leq Ae^{-\lambda t}|v|.$$

- There exist constants $B > 0$ and $\mu > 0$ such that for all $w \in E^{uu}$ and $t > 0$,

$$\|Tf_t(w)\| \geq Be^{\mu t}\|w\|.$$
Bundles $E^{ss}$ and $E^{uu}$ are called the strong stable and strong unstable bundle (or distribution), respectively. From now on we will denote the direct sum $E^{ss} \oplus E^{uu}$ by $E^{su}$. We also define the center (or weak) stable, $E^{ca} = E^c \oplus E^{ss}$ and center (weak) unstable bundle, $E^{cu} = E^c \oplus E^{uu}$; let $X$ denote the vector field tangent to the flow: $X_x = \frac{d}{dt} \big|_{t=0} f_t(x)$. It is well known that all the bundles mentioned above are completely integrable giving rise to continuous foliations: $W^{ss}, W^{uu}, W^{cs}, W^{cu}$. They are called strong stable, strong unstable, center stable and center unstable foliation, respectively.

**Examples** (a) Let $A$ be an $n \times n$ matrix with integer entries and determinant equal to 1. Then $A$, as a linear isomorphism of $\mathbb{R}^n$, preserves the integer lattice $\mathbb{Z}^n$ and therefore projects to a diffeomorphism $h$ of the torus $\mathbb{T}^n$; $h$ is called a linear toral automorphism. If $A$ has no eigenvalues on the unit circle (i.e. if $A$ is hyperbolic), then $h$ is an example of an Anosov diffeomorphism. It turns out that every Anosov diffeomorphism on a torus is topologically conjugate to a linear one. Now let $\{f_t\}$ be the flow on a compact manifold $M$ obtained by suspending $h$. Recall that $M$ is the quotient space of $\mathbb{T}^n \times \mathbb{R}$ by the equivalence relation: $(x, k+1) \sim (h(x), k)$, where $x$ is in $\mathbb{T}^n$ and $k$ is any integer. The flow on $M$ is the projection of the parallel vertical flow on $\mathbb{T}^n \times \mathbb{R}$. It is then not difficult to see that $\{f_t\}$ is an Anosov flow.

(b) Let $S$ be a compact Riemannian manifold with strictly negative (not necessarily constant) curvature, and let $M$ be the unit tangent bundle of $S$. Then the geodesic flow $\{g_t\}$ of $S$ restricted to $M$ is an Anosov flow.

It can be shown that $\{f_t\}$ is not topologically conjugate to $\{g_t\}$, for $\{f_t\}$ has a global cross section and $\{g_t\}$ does not; see, for instance, [Gb].

**Definition 1.2** An Anosov flow is of codimension one if one the the bundles $E^{ss}$, $E^{uu}$ is one dimensional.

From now on, whenever we speak of codimension one Anosov flows we will mean that $E^{uu}$ is one dimensional.
The precise degree of regularity of the distributions above and their corresponding foliations is an important question in the study of Anosov flows. The following theorem follows from the work of Anosov, Hirsch and Pugh, and Hurder and Katok:

**Theorem 1.1 ([An], [HP], [HK])** For a $C^2$ Anosov flow $\{f_t\}$ on a compact manifold $M$ the following are true:

(a) The bundles $E^{ss}, E^{au}, E^{cs}, E^{cu}$ are Hölder continuous.

(b) If $\{f_t\}$ is of codimension one, then $E^{cs}$ is of class $C^{1+\theta}$ for some $\theta$, where $0 < \theta < 1$. Moreover, if also $\dim M = 3$, then the transverse derivative of $E^{cs}$ is of Zygmund class.

(c) If $\{f_t\}$ is of codimension one, preserves a volume form and $\dim M > 3$, then $E^{au}$ is of class $C^1$.

Recall the following definition.

**Definition 1.3** A compact codimension one submanifold $\Sigma$ of a manifold $M$ is called a global cross section for a flow on $M$ if it intersects every orbit of the flow transversely.

Note that we do not require that the orbit of every point of $\Sigma$ intersects $\Sigma$ after some positive time. The reason is that this follows from our definition. Namely, let $x \in \Sigma$. If the orbit of $x$ does not intersect $\Sigma$, then the omega limit set $\omega(x)$ of $x$ also does not intersect $\Sigma$. Let $y \in \omega(x)$. By assumption, the orbit of $y$ intersects $\Sigma$. But $\omega(x)$ is invariant with respect to the flow, and thus must also intersect $\Sigma$. Contradiction. Therefore, the orbit of $x$ returns to $\Sigma$ after some positive time.

If a flow $\{f_t\}$ admits a global cross section $\Sigma$, it can be reconstructed by suspension from the Poincaré (or first return) map $h : \Sigma \to \Sigma$. It is easy to see that if $f_t$ is an Anosov flow, then $h$ is an Anosov diffeomorphism. Thus if an Anosov flow admits a global cross section, then it is topologically conjugate to the suspension of an Anosov diffeomorphism. (Recall that two flows are *topologically conjugate* if there is a homeomorphism taking orbits of one flow onto the orbits of the other, not necessarily preserving the time parameter. Clearly, if the first return time is constant, then the conjugacy can be chosen to preserve the parametrization of the orbits.)
Remark 1.1 Sheldon Newhouse [Nh] and John Franks [Fra] proved that every codimension one Anosov diffeomorphism on a compact manifold is topologically conjugate to a linear toral automorphism. Therefore if a codimension one Anosov flow admits a global cross section, it must be topologically conjugate to the suspension of a linear toral automorphism. Thus the positive answer to the conjecture of Verjovsky (mentioned in the Introduction) would imply a dynamical classification of all codimension one Anosov flows in dimensions greater than three.

We will see that, due to the limited scope of our present methods, regularity of Anosov distributions plays an important role in proving the existence of a global cross section. However, there are methods more topological in nature which also yield similar results. For instance, Plante [Pl3], [Pl4] proved that the conjecture of Verjovsky holds (also when \( \dim M = 3 \)) if and only if \( M \) has solvable fundamental group. Our hope is to unify these two approaches, avoiding the difficult task of dealing with, in general, the low degree of regularity of the strong Anosov distributions.

1.2 Basic Tools for Cross Sections

The question of existence of global cross sections to general flows has been studied extensively; the reader can consult, for instance, [Ch], [Fri], [Sc]. In this section we will only recall two basic tools for proving the existence of global cross sections to Anosov flows, which we will use later.

Theorem 1.2 ([Pl1], Theorem 3.1) Let \( \{ f_i \} \) be an Anosov flow such that \( E^{su} \) is an integrable distribution. Then \( \{ f_i \} \) admits a global cross section.

If \( \alpha \) is a 1-form on \( M \) defined by: \( \text{Ker}(\alpha) = E^{su}, \alpha(X) = 1 \), where \( X \) is the generating vector field, then by Theorem 1.2 and the classical Frobenius theorem on integrability of vector distributions (for a somewhat detailed discussion of this topic the reader should see Section 1.3), we see that if the Stokes exterior differential \( d\alpha \) of \( \alpha \) exists and \( \alpha \wedge d\alpha = 0 \), then \( E^{su} \) is integrable and the flow admits a global section. (In fact, it turns out that if \( d\alpha \) exists in any sense, then \( \alpha \wedge d\alpha = 0 \) if and
only if $d\alpha = 0$.) In [Gh3], E. Ghys showed that if $E^{su}$ is of class $C^1$ and $\{f_t\}$ is of codimension one, then automatically $d\alpha = 0$, hence the flow admits a global cross section. (We will see later that the vanishing of $d\alpha$ is a consequence of a more general fact: namely, if an $n$-manifold $M$ admits a codimension one Anosov flow and $n > 3$, then on $M$ there are no nonzero bounded $f_t$-invariant $k$-forms for $2 \leq k \leq n - 2$.)

In general, the leaves of a foliation $\mathcal{F}$ which integrates $E^{su}$ do not have to be compact; certain conditions that guarantee compactness of leaves of $\mathcal{F}$ can be found in the same paper of Plante [Pl1]. In particular, if $f_t$ is a codimension one Anosov flow, then according to Plante ([Pl1], Theorem 3.7) all the leaves of $\mathcal{F}$ are compact and the conjugacy between $f_t$ and a linear toral automorphism preserves parametrization of orbits.

The other tool for finding cross sections which will be used here also comes from Plante and has to do with the so called joint integrability of the foliations $W^{ss}$ and $W^{uu}$.

**Definition 1.4** We will say that $W^{ss}$ and $W^{uu}$ are jointly integrable if, locally speaking, the projection from one center stable leaf to another along the strong unstable manifolds maps the strong stable leaves to strong stable leaves.

For a more rigorous definition see [Pl1]. Plante showed (see also [Pl1]) that, in fact, the joint integrability of $W^{ss}$ and $W^{uu}$ is equivalent to integrability of $E^{su}$ (see the next section for definition). Therefore, $W^{ss}$ and $W^{uu}$ are jointly integrable if and only if $W^{uu}$ and $W^{ss}$ are.

Now we can state the following useful result.

**Theorem 1.3 ([Pl1], Proposition 1.6)** If $\{f_t\}$ is an Anosov flow such that the foliations $W^{ss}$ and $W^{uu}$ are jointly integrable, then it admits a global cross section.

So, for instance, if $E^{uu}$ is orientable, $C^1$ (as in the case described in part (b) of Theorem 1.1), and $\phi_t$ denotes the flow of some nonsingular $C^1$ vector field in it, then $T\phi_t(E^{ss}) \subset E^{ss}$ (for all $t \in \mathbb{R}$) would imply that $W^{ss}$ and $W^{uu}$ are jointly integrable. (In fact, less is needed; $T\phi_t(E^{ss}) \subset E^{su}$ suffices, as will be shown later.)
1.3 On Integrability of Vector Distributions

Let $M$ be a smooth manifold and $E$ a continuous vector distribution on it (i.e. a continuous section of the Grasmannian of $M$). We will say that $E$ is (uniquely) integrable if: (i) (existence) there is a foliation $\mathcal{F}$ of $M$ whose tangent bundle is equal to $E$; (ii) (uniqueness) any integral manifold of $E$ is a subset of a leaf of $\mathcal{F}$ and, moreover, any smooth curve which at each of its points is tangent to $E$ lies entirely in some leaf of $\mathcal{F}$. (For a detailed discussion of the question of unique integrability of distributions see Section 16 of [An],.) If $E$ is of class $C^1$, then the classical theorem of Frobenius (one of many that bear his name) says that the integrability of $E$ is equivalent to its involutivity: for every two $C^1$ vector fields $X$ and $Y$ in $E$, their Lie bracket $[X,Y]$ is also in $E$. This can also be expressed in terms of differential forms. Namely, if $E$ is of codimension $k$, and $E$ is locally the intersection of the nullspaces of some $C^1$ 1-forms $\omega_1, \ldots, \omega_k$, then the condition for integrability is

$$\omega_1 \wedge \ldots \wedge \omega_k \wedge d\omega_i = 0,$$

for every $1 \leq i \leq k$. By a lemma of Cartan, this is also equivalent to the existence of a continuous 1-form $\eta$ such that $d\omega = \eta \wedge \omega$, where $\omega = \omega_1 \wedge \ldots \wedge \omega_k$.

However, there are plenty of examples of distributions which are not $C^1$ but are nevertheless integrable; such are the strong stable and strong unstable distributions of any Anosov flow. Clearly, in the case of such distributions we cannot talk about involutivity in the classical sense. This question was addressed by P. Hartman in his book [Ha]. First recall that a differential $k$-form $\alpha$ on an open set $U$ is said to be differentiable in the Stokes sense if there exists a continuous $(k+1)$-form $\beta$ such that for every immersed $C^1$ $(k+1)$-ball $D$ in $U$ with $\partial D$ piecewise $C^1$,

$$\int_{\partial D} \alpha = \int_D \beta.$$

Then $\beta$ is called the Stokes exterior differential of $\alpha$. Hartman shows that a continuous distribution $E$ is integrable if and only if $\omega_i$’s (defined as above) are differentiable in the Stokes sense, and their Stokes exterior differentials, $d\omega_i$, satisfy the relation (1.1).

Unfortunately, Stokes differentiability is not easy to verify. Hartman shows that a continuous form $\alpha$ is Stokes differentiable if and only if it is locally the uniform
limit of a sequence \((\alpha_j)\) of \(C^1\) forms such that \(d\alpha_j\) is locally uniformly convergent ([Ha], Lemma 5.1 on page 102). However this does not relate the notion of Stokes differentiability of a non-\(C^1\) form to any other notion of regularity which would be easy to check.

That is why we take a different approach, more suited to our purposes. Our goal is to generalize the theorem of Frobenius to (locally) Lipschitz distributions. We deal with ordinary exterior differentials (in fact, for the reader familiar with Whitney’s book [Wh], it will not be difficult to see that Lipschitz forms are “flat”, hence, by Whitney, Stokes differentiable and their Stokes differential coincides with the ordinary one in Whitley’s sense), but we show that it is enough for the distribution in question to be involutive almost everywhere (which is equivalent with (1.1) holding a.e.).

First let us recall the main notions.

Let \(M\) be a \(C^\infty\) \(n\)-dimensional Riemannian manifold.

**Definition 1.5** We will say that a distribution (or plane field) \(E\) on \(M\) is Lipschitz if it is locally spanned by Lipschitz continuous vector fields.

Recall that a map \(f\) between metric spaces \((M_1, d_1)\) and \((M_2, d_2)\) is called Lipschitz continuous (or simply Lipschitz) if there is a constant \(C > 0\) such that \(d_2(f(p), f(q)) \leq C d_1(p, q)\), for all \(p, q \in M_1\). By saying that a vector field \(X\) on \(M\) is Lipschitz we mean that in some (and therefore in any) smooth coordinate system, \(X\) can be written in the form

\[
X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i},
\]

where each \(a_i\) is a Lipschitz function. Recall that a Lipschitz vector field is uniquely integrable and each time \(t\) map of its flow is Lipschitz.

An important property of Lipschitz maps, discovered by H. Rademacher, is that they are almost everywhere (relative to Lebesgue measure) differentiable in the ordinary sense with locally essentially bounded derivative. This property enables us to extend the notion of Lie bracket to Lipschitz vector fields. Namely, if \(X, Y\) are Lipschitz vector fields, and \(f\) is a \(C^\infty\) function, we define

\[
[X, Y]f = X(Yf) - Y(Xf).
\]
Note that this expression makes sense and is defined a.e. because $Xf$ and $Yf$ are Lipschitz functions. If in some local coordinates $X$ and $Y$ can be expressed as:

$$X = \sum_i a_i \frac{\partial}{\partial x_i}, \quad Y = \sum_j b_j \frac{\partial}{\partial x_j},$$

then $[X, Y]$ in the same coordinate system looks like this:

$$[X, Y] = \sum_{i,j} \left( a_i \frac{\partial b_j}{\partial x_i} - b_j \frac{\partial a_i}{\partial x_j} \right) \frac{\partial}{\partial x_j}.$$

Notice that the coefficients of $[X, Y]$ are locally bounded functions.

**Definition 1.6** Let $E$ be a Lipschitz distribution on $M$. We will say that $E$ is involutive a.e. if for every two Lipschitz vector fields $X, Y$ in $E$ their bracket $[X, Y]_p$ belongs to $E_p$ for a.e. $p \in M$.

So, for instance, if on some open set $X_1, \ldots, X_k$ is a local basis for $E$ consisting of Lipschitz vector fields, then $E$ is involutive a.e. if and only if there exist locally bounded functions $c_{ij}$ such that almost everywhere

$$[X_i, X_j] = \sum_{i=1}^{k} c_{ij}X_i.$$

Now we can state the Frobenius theorem for Lipschitz distributions.

**Theorem 1.4** Let $E$ be a $k$-dimensional Lipschitz distribution on a compact smooth $n$-dimensional manifold $M$. If $E$ is involutive almost everywhere, then every point of $M$ has a coordinate neighborhood $(U; x_1, \ldots, x_n)$ such that:

(a) Each map $x_i : U \to \mathbb{R}$ is Lipschitz.

(b) The slices $x_{k+1} = \text{constant}, \ldots, x_n = \text{constant}$ are integral manifolds of $E$.

Moreover, every connected integral manifold of $E$ in $U$ is of class $C^{1,\text{Lip}}$ and lies in one if these slices.

Furthermore,

**Theorem 1.5** Let $E$ be as above. Then through every point $p$ of $M$ passes a unique maximal connected integral manifold of $E$, and every connected integral manifold of $E$ through $p$ is contained in the maximal one.
Corollary 1.5.1 Let $\alpha$ be a 1-form on $M$ which is everywhere nonsingular and Lipschitz, and let $E = \text{Ker}(\alpha)$. Then $E$ is uniquely integrable if and only if

$$\alpha \wedge d\alpha = 0$$

almost everywhere on $M$.

Proof of Theorem 1.4

The proof is by induction on $k$. In constructing the desired coordinate system we closely follow Warner (see [Wa], Theorem 1.60), but when certain difficulties arise due to nonsmoothness of $E$, we use some standard approximation techniques to reach the desired conclusions.

If $k = 1$, the theorem follows directly from the flowbox theorem and the already mentioned fact that Lipschitz vector fields generate Lipschitz flows.

So assume $k \geq 2$ and that the theorem holds for $k - 1$. Given $p \in M$, let $(V; y_1, \ldots, y_n)$ be a $C^\infty$ coordinate neighborhood with $y_i(p) = 0$ $(1 \leq i \leq n)$, on which $E$ is spanned by Lipschitz vector fields $X_1, \ldots, X_k$. Without loss of generality we may assume that $X_1(y_1) \geq 1$ on $V$. Set $Y_1 = X_1$ and

$$Y_i = X_i - \frac{X_i(y_1)}{X_1(y_1)}X_1,$$

for $2 \leq i \leq k$. Then $Y_1, \ldots, Y_k$ are linearly independent Lipschitz vector fields on $V$ spanning $E|_V$. Let $H$ be the slice $y_1 = 0$ and let

$$Z_i = Y_i|_H,$$

for $2 \leq i \leq k$. Then, by construction, $Y_i(y_1) = 0$ for $2 \leq i \leq k$. So the vector fields $Z_2, \ldots, Z_k$ are tangent to $H$ and span a $(k - 1)$-dimensional Lipschitz distribution $F$ on $H$. We claim that $F$ is involutive a.e.. To see this, let $\iota : H \rightarrow M$ be the inclusion. Then $Y_i = \iota_* (Z_i)$ so

$$[Y_i, Y_j] = \iota_*([Z_i, Z_j]).$$

Observe that for $i, j \geq 2$,

$$[Y_i, Y_j](y_1) = Y_i(Y_jy_1) - Y_j(Y_iy_1) = 0.$$
On the other hand, because of the involutivity of $E$, there exist locally bounded functions $c_{ij}^l$ such that

$$[Y_i, Y_j] = \sum_{l=1}^{k} c_{ij}^l \cdot Y_i$$  \hspace{1cm} (1.2)$$
a.e. on $U$. If $i \geq 2$ and $j \geq 2$, then according to the observation above: $c_{ij}^1 = 0$, that is,

$$[Y_i, Y_j] = \sum_{l=2}^{k} c_{ij}^l Y_i.$$

Thus, since $\iota_s$ is 1-1, we have:

$$[Z_i, Z_j] = \sum_{l=2}^{k} c_{ij}^l \bigg|_H \cdot Z_l$$
a.e. on $H$ (with respect to the $(n - 1)$-dimensional Lebesgue measure on $H$). So $F$ is involutive a.e.. By the induction hypothesis, there exists a coordinate neighborhood $(U; z_2, \ldots, z_n)$ of $p$ with $U \subset V$, such that the slices

$$z_{k+1} = constant, \ldots, z_n = constant$$

are precisely the integral manifolds of $F$.

Let $\phi_t$ be the local flow of $X_1 (= Y_1)$ on $U$. There exists a neighborhood of $p$ which we (to simplify notation) also call $U$, such that the projection $\pi : U \rightarrow H \cap U$ along the orbits of $\phi_t$ is well defined and Lipschitz.

Now define maps from $U$ to $\mathbb{R}$ by:

$$x_1(q) = t,$$

$$x_j = z_j \circ \pi,$$

where $2 \leq j \leq n$ and $x_1(q) = t$ if and only if $\phi_{-t}(q) \in H \cap U$. It is clear that $(U; x_1, \ldots, x_n)$ is a Lipschitz coordinate system.

So far our proof differs very little from the proof presented in [Wa]. From this point on we start dealing with fine properties of Lipschitz functions using the standard method of “mollification”.

Thus it remains to show that

$$Y_i(x_j) = 0$$
a.e. on $U$ for $1 \leq i \leq k$ and $k + 1 \leq j \leq n$. First let us approximate the functions $x_j$ by smooth ones. Since the statements above are of local character, without loss of generality we may assume that we are in $\mathbb{R}^n$ where we have the standard mollifiers $\eta_\varepsilon$ at our disposal (see [EG] or [Zil]). Let $x_j^\varepsilon = x_j * \eta_\varepsilon$, where $*$ denotes convolution. Then:

(i) Each $x_j^\varepsilon$ is of class $C^\infty$.

(ii) As $\varepsilon \to 0$, $x_j^\varepsilon \to x_j$ uniformly on compact sets.

(iii) As $\varepsilon \to 0$, $D^\alpha x_j^\varepsilon \to D^\alpha x_j$ in $L^1_{\text{loc}}$ and also pointwise almost everywhere, for every “multiindex” $\alpha$, $|\alpha| = 1$.

Furthermore, by the properties of convolution:

$$Y_i(x_j^\varepsilon) = (Y_i x_j) * \eta_\varepsilon$$

$$= 0,$$

for all $2 \leq j \leq n$ and small $\varepsilon > 0$. Therefore $Y_i(Y_i x_j^\varepsilon) = [Y_i, Y_i](x_j^\varepsilon)$ ($j \geq 2$). By (1.2), we have:

$$Y_i(Y_i x_j^\varepsilon) = \sum_{i=2}^k c_{ii} \cdot Y_i x_j^\varepsilon$$

(1.3)

almost everywhere. Since $Y_i$ is Lipschitz, the foliation of $U$ by orbits of $\phi_i$ is absolutely continuous which implies that for a.e. $q \in H$ (with respect to the $(n - 1)$-dimensional Lebesgue measure on $H$) the $\phi_i$-orbit of $q$ intersects any set of set of $n$-dimensional Lebesgue measure zero along a set of 1-dimensional measure zero. So integration of both sides of (1.3) along $\phi_s(q)$, $0 \leq s \leq t$, yields

$$(Y_i x_j^\varepsilon)(\phi_t(q)) - (Y_i x_j^\varepsilon)(q) = \int_0^t \sum_{i=2}^k c_{ii} \cdot (\phi_s(q)) (Y_i x_j^\varepsilon) (\phi_s(q)) \, ds,$$  

(1.4)

for a.e. $q \in H$ and a.e. $t \in J(q)$, where $J(q)$ is some open interval in $\mathbb{R}$ depending on $q$.

Now let $\varepsilon \to 0$. By (i)-(iii), we obtain:

$$(Y_i x_j)(\phi_t(q)) - (Y_i x_j)(q) = \int_0^t \sum_{i=2}^k c_{ii} \cdot (\phi_s(q)) (Y_i x_j)(\phi_s(q)) \, ds,$$

(1.5)
for a.e. \( q \in H \) and a.e. \( t \in J(q) \). Since for \( 2 \leq i \leq k \) the vector fields \( Y_i |_H (= Z_i) \) belong to the distribution \( F \), for \( k + 1 \leq j \leq n \) we have
\[
(Y_i x_j)(q) = (Z_i x_j)(q) = 0
\]
a.e., because the slices \( z_j = constant \ (k + 1 \leq j \leq n) \) are integral manifolds of \( F \).

Fix \( q \in H \) so that (1.5) holds for a.e. \( t \in J(q) \). (There will be a set of full measure of such \( q \).) The right hand side of (1.5) is a continuous function of \( t \). Therefore the functions \( t \mapsto (Y_i x_j)(\phi_t q) \) \( (j \) fixed, \( k + 1 \leq j \leq n) \) are a.e. continuous and satisfy the following \( (k - 1) \times (k - 1) \) homogeneous system of linear integral equations (along the orbit of \( q \)) with \( L^\infty \) coefficients:
\[
Y_i x_j = \int_0^t \sum_{i=2}^n C_{ij} Y_i x_j \, ds. \quad (1.6)
\]
Let \( C(t) \) be the matrix with entries \( c_{ij}(\phi_t q) \) and let
\[
f(t) = (Y_2 x_j(\phi_t(q)), \ldots, Y_k x_j(\phi_t(q))).
\]
Then (1.6) becomes
\[
f(t) = \int_0^t C(s) \cdot f(s) \, ds.
\]
So
\[
\|f(t)\| \leq \int_0^t \|C(s)\| \cdot \|f(s)\| \, ds.
\]
As remarked above, \( f \) is continuous a.e. and \( |C| \in L^\infty \). Gronwall’s inequality (see [Hi], Theorem 1.5.7) implies \( f(t) = 0 \) a.e..

This proves that \( Y_i x_j = 0 \) a.e.. It remains to show that the slices
\[
x_{k+1} = constant, \ldots, x_n = constant
\]
are integral manifolds of \( E \).

Let \( S \) be one such slice. Then its tangent bundle can be expressed as:
\[
T S = \bigcap_{j=k+1}^n Ker(dx_j |_S),
\]
which clearly contains the vector fields $Y_1|_S, \ldots, Y_k|_S$ a.e.. Thus $T_qS = E_q$ a.e. on $S$. Since $E$ is a continuous distribution defined on a compact space, we can extend this relation over all of $S$. Thus $S$ is an integral manifold of $E$. Note also that since the tangent bundle of $S$ is Lipschitz, it follows that $S$ is a manifold of class $C^{1,Lip}$.

This completes the first part of the proof.

It remains to show uniqueness of integral manifolds.

Let $N$ be a connected integral manifold of $E$ in $U$. Denote the projection $\mathbb{R}^n \to \mathbb{R}^{n-k}$ to the last $n-k$ coordinates by $pr$. If $\varphi = (x_1, \ldots, x_n)$ is the coordinate map defined above, then

$$T(pr \circ \varphi)(TN) = T(pr \circ \varphi)(E|_N)$$

$$= 0$$

a.e. on $N$. Since $N$ is connected, it follows (see e.g. [Zi]) that $pr \circ \varphi = const$ a.e. on $N$. By continuity, this equality holds everywhere on $N$, which implies that $N$ is contained in a slice $x_{k+1} = const, \ldots, x_n = const$.

This completes the proof. $\square$
Proof of Theorem 1.5 and Corollary 1.5.1

The proof of existence and uniqueness of maximal integral manifolds in the classical case given in [Wa] (Theorem 1.64), is valid in our setting.

To prove Corollary 1.5.1, choose a continuous vector field $X$ such that $\alpha(X) = 1$. Let $Y$ and $Z$ be arbitrary Lipschitz vector fields in $E$. Then

$$(\alpha \wedge d\alpha)(X, Y, Z) = \alpha(X) \, d\alpha(Y, Z)$$

$$= d\alpha(Y, Z)$$

$$= Y(\alpha(Z)) - Z(\alpha(Y)) - \alpha([Y, Z])$$

$$= -\alpha([Y, Z]).$$

(A remark is in place here. The third equality above is well known for $C^1$ forms. However, since Lipschitz forms are nicely approximable by smooth forms, as demonstrated above, this equality continues to hold almost everywhere for Lipschitz forms.)

So if $\alpha \wedge d\alpha = 0$ a.e., then $[Y, Z]_p \in E_p$ for a.e. $p \in M$, and conversely, if $E$ is a.e. closed under bracket, then $\alpha \wedge d\alpha$ is zero a.e. The Corollary now follows from Theorems 1.4 and 1.5. □

1.4 Anosov Flows for which $E^{stu}$ is Lipschitz

In this section we use the results of the previous one to show that the conjecture of Verjovskij holds if the distribution $E^{stu}$ is Lipschitz. We also prove a similar result for Anosov flows of any codimension.

First let us recall that if $A$ is a linear isomorphism of normed vector spaces, the conorm (or minimum norm) of $A$ is defined to be

$$m(A) = \inf\{|Av| : |v| = 1\}.$$ 

Now we can state the main results of this chapter.
Theorem 1.6 Let \( \{f_i\} \) be an Anosov flow on a compact manifold \( M \) such that \( E^{su} \) is Lipschitz and
\[
\mu := \inf_{x \in M} \left[ m(T_x f_\tau | E^{su}) \cdot m(T_x f_\tau | E^{us}) \right] > 1,
\]
for some \( \tau > 0 \). Then \( \{f_i\} \) admits a global cross section.

Theorem 1.7 Let \( \{f_i\} \) be a codimension one Anosov flow on a compact manifold \( M \) of dimension \( n > 3 \). If \( E^{su} \) is Lipschitz, then \( \{f_i\} \) admits a global cross section with constant return time.

Note that Theorem 1.7 implies that lipschitzness of \( E^{su} \) is extremely fragile: it can be destroyed by any \( C^1 \) small non-constant time reparametrization of the flow.

Proof of Theorem 1.6

We will need the following generalization of Lemma 1.2 from [Gh3].

Lemma 1.1 Let \( E_i \) \((i = 1, 2)\) be Euclidean spaces and assume the splitting \( E_i = S_i \oplus U_i \) is orthogonal \((i = 1, 2)\). Let \( f : E_1 \to E_2 \) be a linear isomorphism such that \( f(S_1) = S_2 \) and \( f(U_1) = U_2 \).

(a) If \( \mu := m(f|_{S_1}) m(f|_{U_1}) \), then for all \( w_s \in S_2 \) and \( w_u \in U_2 \):
\[
\|f^{-1}(w_s \wedge w_u)\| \leq \frac{1}{\mu} \|w_s \wedge w_u\|.
\]

(b) If \( \dim E_i = n - 1 \), \( \dim U_1 = 1 \) and \( \|f(w)\| \leq \nu \|w\| \), for some \( \nu > 0 \) and all \( w \in S_1 \), then
\[
(\det f) \|f^{-1}(w_s \wedge w_u)\| \leq \nu^{n-3} \|w_s \wedge w_u\|
\]
for all \( w_s \in S_2 \), \( w_u \in U_2 \).

Proof (a) Let \( v_s \in S_1 \), \( v_u \in U_1 \) be arbitrary. Then
\[
\|f(v_s \wedge v_u)\| = \|f(v_s)\| \|f(v_u)\| \\
\geq m(f|_{S_1}) m(f|_{U_1}) \|v_s\| \|v_u\| \\
= \mu \|v_s \wedge v_u\|.
\]
Part (a) now follows if we take \( w_s = f(v_s) \) and \( w_u = f(v_u) \).

(b) Let \( v_s, v_u \) be as above. Choose unit vectors \( e_3, \ldots, e_{n-1} \) in \( S_1 \) such that \( e_1 = v_u, e_2 = v_s, e_3, \ldots, e_{n-1} \) is an orthogonal basis of \( E_1 \). Then we have:

\[
\|v_u \wedge v_s\| \cdot \det f = \left| f(e_1 \wedge \cdots \wedge e_{n-1}) \right|
\leq \left| f(v_u \wedge v_s) \right| \prod_{i=3}^{n-1} \left| f(e_i) \right|
\leq \nu^{n-3} \left| f(v_u \wedge v_s) \right|.
\]

To complete the proof, take \( w_u = f(v_u), w_s = f(v_s) \). \( \square \)

Let us now prove Theorem 1.6. Define a 1-form \( \alpha \) by requiring that

\[
\text{Ker}(\alpha) = E^{su}, \quad \alpha(X) = 1,
\]

where \( X \) is the vector field that generates the flow. Since \( E^{su} \) is a Lipschitz distribution, \( \alpha \) is a Lipschitz form, so \( d\alpha \) exists on an \( f_t \)-invariant set of full measure in \( M \). Clearly, \( f^*_t \alpha = \alpha \) for all \( t \in \mathbb{R} \). Since \( d\alpha \) is the ordinary exterior differential, it commutes with pullbacks by diffeomorphisms\(^1\) and it follows that \( f^*_t (d\alpha) = d\alpha \) for all \( t \in \mathbb{R} \).

Let \( x \in M \) be a point where \( d\alpha \) is defined, and let \( w_s \in E^{ss} \) and \( w_u \in E^{su} \) be arbitrary. Part (a) of the Lemma implies that \( \|(f_{-\tau})_*(w_s \wedge w_u)\| \leq \mu^{-1} \|w_s \wedge w_u\| \). Therefore,

\[
|d\alpha(w_s, w_u)| = |f^*_k (d\alpha)(w_s, w_u)|
= |d\alpha((f_{-\tau})_*(w_s \wedge w_u))|
\leq |d\alpha|_\infty \mu^{-k} \|w_s \wedge w_u\|
\to 0,
\]

as \( k \to \infty \). (Here \( |d\alpha|_\infty \) denotes the \( L^\infty \) norm of \( d\alpha \). Since \( \alpha \) is Lipschitz and \( M \) is compact, this norm is finite.) Similarly we can show that \( d\alpha_x(X, v) = 0 \) for almost every \( x \in M \), where \( v \in E^{su} \). (In fact, we don’t need the lemma to prove this. For

\(^1\)This follows simply by the Chain Rule.
details see [Gh3]. Therefore \( d\alpha = 0 \) a.e. so by theorems 1.6 and 1.7 it follows that
\( E^{su} \) is an integrable distribution. By the already mentioned result of Plante, the flow admits a global cross section. \( \square \)

**Proof of Theorem 1.7**

First note that if \( f_t \) preserves a volume form on \( M \), then Theorem 1.7 follows directly from Theorem 1.6; higher dimensionality of \( M \) is strongly used here. Namely, if \( n > 3 \) and \( f_t \) is a volume preserving codimension one Anosov flow with \( \dim E^{uu} = 1 \), then

\[
m(T_x f_t |_{E^{ss}}) m(T_x f_t |_{E^{uu}}) = \mu_1 \lambda > \mu_1 \mu_2 \cdots \mu_{n-2} \lambda = \det T_x f_t = 1,
\]

for every \( x \in M \). Here \( \mu_i < 1 \) are the contraction rates of \( T_x f_t \) on \( E^{ss} \), and \( \lambda > 1 \) is the expansion rate of \( T_x f_t \) on \( E^{uu} \). Since \( M \) is compact, the strict inequality above also holds for the infimum over all \( x \in M \).

If \( f_t \) is not volume preserving, denote by \( \Delta(x, t) \) the determinant of \( T_x f_t \). Define the 1-form \( \alpha \) as above and let \( \nu = \sup_{x \in M} \| T_x f_t |_{E^{ss}} \| \). Clearly, \( \nu < 1 \).

Assume that \( E^{uu} \) is 1-dimensional and let \( Y \) be a unit continuous vector field in \( E^{uu} \) (it is no loss of generality to assume that all the bundles are orientable) and let \( Z \) be a continuous, not necessarily nonvanishing vector field in \( E^{ss} \). Define a function \( h : M \to \mathbb{R} \) by

\[
h(x) = d\alpha_x(Y_x, Z_x).
\]

Then \( h \in L^\infty(M) \) and:

\[
\int_M |h(x)| \, dx = \int_M \Delta(x, t) |h(f_t x)| \, dx = \int_M \Delta(x, t) |d\alpha_{f_t x}(Y_{f_t x}, Z_{f_t x})| \, dx = \int_M \Delta(x, t) |(f^*_t d\alpha)_{f_t x}(Y_{f_t x}, Z_{f_t x})| \, dx
\]
\[
\begin{align*}
&= \int_M \Delta(x, t) \left| (d\alpha)_x ((f_{-t})_*(Y \wedge Z)) \right| \, dx \\
&\leq |d\alpha|_\infty \int_M \Delta(x, t) \| (f_{-t})_*(Y \wedge Z) \| \, dx \\
&\leq C \int_M \nu^{(n-3)\mu} \, dx \\
&= C \text{volume}(M) \nu^{(n-3)\mu} \\
\rightarrow 0, 
\end{align*}
\]

as \( t \to \infty \). (Inequality (1.8) follows from part (b) of the Lemma.) Therefore \( h(x) = 0 \) a.e.. Since \( d\alpha(X, v) = 0 \) a.e. for \( v \in E^{su} \), and since \( Z \) was arbitrary, it follows that \( d\alpha = 0 \) a.e., which completes the proof. \( \square \)
Chapter 2

More on Cross Sections to Anosov Flows

The main result of this chapter is that volume preserving codimension one Anosov flows on manifolds of dimension greater than three for which the distribution $E^{su}$ is of class $\text{Lip}_-\text{h}$ have cross sections. Thus we weaken the regularity assumption on $E^{su}$ but pay the price by requiring that the flow be volume preserving.

We recall that a function between metric spaces is said to be of class $\text{Lip}_-\text{h}$ if it is $\theta$-Hölder continuous for all $0 < \theta < 1$. By saying that a distribution $E$ on a Riemannian manifold $M$ is of class $\text{Lip}_-\text{h}$ we mean that it is $\text{Lip}_-\text{h}$ as a section of the Grassmannian of $M$.

Before we state and prove the main theorem, we need to discuss some properties of the expansion cocycle of a codimension one Anosov flow.

2.1 The Expansion Cocycle

Let $\{f_t\}$ be codimension one Anosov flow on a compact manifold $M$. Without loss of generality we assume that all the Anosov distributions are orientable (otherwise, we pass to an appropriate finite cover of $M$). Choose a continuous Riemann structure $\langle \cdot, \cdot \rangle$ on $M$ such that:

- $TM = E^{uu} \oplus E^{cs}$ is an orthogonal splitting.
• $X$ (the Anosov vector field) has length 1.

Let $Z$ be a nonvanishing $C^\infty$ vector field on $M$ which is everywhere transverse to $E^{cs}$. Let $Y$ be the projection of $Z$ on $E^{uu}$ relative to the chosen Riemannian metric. Then $Y$ is a continuous, completely integrable nonvanishing vector field. If necessary, modify the metric in a continuous fashion to make $Y$ a unit vector field and define a positive function $\lambda : M \times \mathbb{R} \to \mathbb{R}$ by

$$T_{x,f_t}(Y_x) = \lambda(x,t) Y_{f_t x},$$

with $x \in M$ and $t \in \mathbb{R}$. We call $\lambda$ the expansion cocycle of $f_t$ (relative to the chosen Riemannian metric). Clearly, $\lambda$ is continuous in $x$ (we will show more later), $C^1$ in $t$, $\lim_{t \to +\infty} \lambda(x,t) = \infty$, and $\lim_{t \to -\infty} \lambda(x,t) = 0$ uniformly in $x \in M$. In fact, there exist numbers $K > 0$ and $c > 0$ such that for all $x \in M$ and $t \geq 0$,

$$\lambda(x,t) \geq Ke^ct.$$

Define a 1-form $\omega$ by

$$Ker(\omega) = E^{cs}, \quad \omega(Z) = 1.$$

Since $E^{cs}$ is of class $C^1$, so is $\omega$. (That is the only reason we needed $Z$: to ensure that $\omega$ is smooth in a direction transverse to $E^{cs}$.) Note also that $\omega(Y) = 1$. (This is because $Z = Y + V$ where $V \in E^{cs}$ and $\omega(V) = 0$.) By Frobenius theorem, $\omega$ divides $d\omega$, i.e. there exists a continuous 1-form $\eta$ such that $d\omega = \eta \wedge \omega$. Define a function

$$u : M \to \mathbb{R}$$

by

$$u(x) = \eta(X_x).$$

Then we have the following

**Theorem 2.1** (a) The function $u$ is of class $C^1$.

(b) For all $x \in M$ and $t \in \mathbb{R}$ we have:

$$u(f_t x) = \left. \frac{d}{ds} \right|_t \log \lambda(x, s).$$

Therefore:

$$\lambda(x,t) = \exp \left\{ \int_0^t u(f_s x) \, ds \right\}. \quad (2.1)$$
(c) Let $c$ and $K$ be as above. Then for every point $x \in M$ and every $\tau > 0$,

$$\frac{1}{\tau} \int_0^\tau u(f_\tau x) dt \geq \frac{1}{\tau} \log K + c.$$

**Proof** (a) We have

$$u = \eta(X)$$

$$= \eta(X)\omega(Z) - \eta(Z)\omega(X)$$

$$= (\eta \wedge \omega)(X, Z)$$

$$= d\omega(X, Z)$$

$$= X(\omega(Z)) - Z(\omega(X)) - \omega([X, Z])$$

$$= -\omega([X, Z])$$

because $\omega(X) = 0$ and $\omega(Z) = 1$. Since both $[X, Z]$ and $\omega$ are of class $C^1$, part (a) follows.

(b) Note that $\omega(Tf_t(Z_x)) = \lambda(x, t)$. Using the properties of the Lie derivative (see, for instance, [Wa]), we obtain:

$$u(x) = d\omega(X, Z)$$

$$= i_Z i_X d\omega$$

$$= i_Z (i_X d + d i_X)\omega$$

$$= i_Z (L_X \omega)$$

$$= (L_X \omega)(Z)$$

$$= \frac{d}{dt} \bigg|_0 [\omega(Tf_t(Z_x))]$$

$$= \frac{d}{dt} \bigg|_0 \lambda(x, t).$$

Clearly, $\lambda(x, t)$ is a multiplicative 1-cocycle over the Anosov flow, that is:

$$\lambda(x, t + s) = \lambda(x, t) \lambda(f_i(x), s). \quad (2.2)$$

Differentiating the last equality with respect to $s$ at zero, we obtain

$$u(f_i x) = \frac{d}{ds} \bigg|_t \log \lambda(x, s).$$
This clearly implies formula (2.1).

(c) Let \( x \) be an arbitrary point and \( \tau > 0 \). Then by (b):

\[
\int_0^\tau u(f_s x) dt = \log \lambda(x, \tau) - \log \lambda(x, 0) \\
= \log \lambda(x, \tau) \\
\geq \log (K e^{\tau}) \\
= \log K + \tau c,
\]

as desired. We remark that this could be used to construct an everywhere positive function \( u' \) with analogous properties as those of \( u \), but we postpone this until Chapter 3. □

**Corollary 2.1.1** There exists a constant \( \ell > 0 \) such that if \( y \in W^{au}(x) \), then for all \( t > 0 \):

\[
\exp\{-\ell d_u(x, y)\} \leq \frac{\lambda(x, -t)}{\lambda(y, -t)} \leq \exp\{\ell d_u(x, y)\}.
\]

Here \( d_u(x, y) \) denotes the distance between \( x \) and \( y \) along \( W^{au}(x) \).

**Proof** Suppose \( y \in W^{au}(x) \). Then \( d(f_{-s} x, f_{-s} y) \leq A e^{-\mu s} d(x, y) \) for some \( A, \mu > 0 \) and all \( s > 0 \). Let \( B = \sup_M \|du\| \). If \( t > 0 \), then (2.1) implies:

\[
\frac{\lambda(x, -t)}{\lambda(y, -t)} = \exp \left\{ \int_0^{-t} [u(f_s x) - u(f_s y)] ds \right\} \\
\leq \exp \left\{ \int_0^{t} [u(f_{-s} x) - u(f_{-s} y)] ds \right\} \\
\leq \exp \left\{ B \int_0^{t} d(f_{-s} x, f_{-s} y) ds \right\} \\
\leq \exp \left\{ A B \int_0^{t} e^{-\mu s} d_u(x, y) ds \right\} \\
= \exp \left\{ -\frac{A B d_u(x, y)}{\mu} (e^{-\mu t} - 1) \right\} \tag{2.3}
\]

where \( \ell = AB/\mu \). Observe that (2.3) is increasing with respect to \( t \). The \( \geq \) part of the inequality follows by switching the roles of \( x \) and \( y \). □
It is easy to see that our reasoning used in the proof of Corollary 2.1.1 and Theorem 2.1 also proves the following result.

**Theorem 2.2** Let \( \{f_t\} \) be an Anosov flow whose center stable distribution is of class \( C^\theta \) for some \( \theta \leq 1 \). Let \( \lambda(x, t) \) be the determinant of \( T_xf_t \) on \( E^{uu} \). Then there exists an \( \ell > 0 \) such that for all \( x, y \in M \) with \( y \in W^{uu}(x) \), and all \( t > 0 \),

\[
\exp \{-\ell d_u(x, y)^\theta\} \leq \frac{\lambda(x,-t)}{\lambda(y,-t)} \leq \exp \{\ell d_u(x, y)^\theta\}
\]

Therefore, there is some uniformity in the behavior of \( x \mapsto \lambda(x, t) \) along the strong unstable manifolds of the flow. It is natural to ask the following question: can the constant \( \ell \) be made as close to 0 as we want by suitably reparametrizing the flow? Or even better: can the given flow be reparametrized to make the expansion cocycle of the new flow independent of \( x \)?

These questions will be fully answered in Chapter 3.

### 2.2 Anosov Flows for which \( E^{su} \) Is Lip–

Now we are ready to state the main theorem of this chapter.

**Theorem 2.3** Let \( \{f_t\} \) be a volume preserving \( C^2 \) codimension one Anosov flow on a compact manifold \( M \) of dimension \( n > 3 \). If the distribution \( E^{su} \) is of class Lip–, then \( \{f_t\} \) admits a global cross section.

**Proof of Theorem 2.3**

By Theorem 1.1 we know that \( E^{uu} \) is of class \( C^1 \). (Without loss of generality we may assume that it is also orientable.) Let \( Y \) and \( \lambda \) be as in Section 2.1. In our case, \( Y \) is \( C^1 \) and we can also choose a \( C^1 \) Riemann structure satisfying the same properties as those we required in the previous section. Let \( \{\phi_t\} \) be the flow of \( Y \). It is easy to see that \( \phi_t \) is of class \( C^1 \).

Let \( \alpha \) be a 1-form on \( M \) defined, as before, by

\[
\text{Ker}(\alpha) = E^{su}, \quad \alpha(X) = 1,
\]
where $X$ is the Anosov vector field. Since $E^{su}$ is of class Lip$^-$, so is $\alpha$. Moreover, $\alpha$ is invariant with respect to $f_t$, that is, $f_t^* \alpha = \alpha$ for all $t \in \mathbb{R}$.

Now let $v$ be an arbitrary vector in $E^{ss}$. There exist continuous functions $a_v : \mathbb{R} \to \mathbb{R}$ and $b_v : \mathbb{R} \to \mathbb{R}$ such that for all $t \in \mathbb{R}$:

$$T\phi_t(v) = a_v(t) Y + b_v(t) X + Z_v(t),$$

where $Z_v(t) \in E^{ss}$. We will show that $b_v = 0$ for all $v \in E^{ss}$, by analyzing the behavior of $b_v(t)$ under iteration of the Anosov flow on the vector $v$. For that we need an estimate of the size of $b_v(t)$ for small $t$.

**Lemma 2.1** Let $b_v$ be as above. Then for every $0 < \theta < 1$ there exists a constant $C > 0$, independent of $v$, such that for all $|t| \leq 1$:

$$|b_v(t)| \leq C |t|^\theta \|v\|.$$

**Proof** Let $\bar{d}$ denote the distance function on $TM$ (the tangent bundle of $M$) induced by some Riemannian metric on its tangent bundle. Since $\alpha$ is Lip$^-$, for every $0 < \theta < 1$ and every $\epsilon > 0$ there exists a constant $A(\theta, \epsilon) > 0$ such that

$$|\alpha(u) - \alpha(w)| \leq A(\theta, \epsilon) \bar{d}(u, w)^\theta,$$

(2.4)

for all $u, w \in TM$ for which $\bar{d}(u, w) \leq \epsilon$. (Note that on noncompact spaces like $TM$, Hölderiness is a condition valid only on a “small scale”.) Also there exists an $\epsilon_0 > 0$ such that for every $v \in E^{ss}$ with norm $\leq 1$ and every $|t| \leq 1$,

$$\bar{d}(T\phi_t(v), v) \leq \epsilon_0.$$

Let $0 < \theta < 1$ be arbitrary and set $B = A(\theta, \epsilon_0)$. Then for all $|t| \leq 1$ and $v \in E^{ss}$, $\|v\| \leq 1$, we have:

$$|b_v(t)| = |\alpha(T\phi_t(v))|$$

$$= |\alpha(T\phi_t(v)) - \alpha(v)|$$

$$\leq B \bar{d}(T\phi_t(v), v)^\theta$$

(2.5)
\[
\begin{align*}
\leq B \left| \int_0^t \frac{d}{ds} \left[ T_x \phi_s(v) \right] \| ds \right|^\theta \\
= B \left| \int_0^t \left[ (T_{\phi_s} Y)(T_x \phi_s)(v) \right] \| ds \right|^\theta \\
\leq C |t|^\theta \| v \|^\theta,
\end{align*}
\]

(2.6) (2.7) (2.8)

where \( C = B (\sup_M |TY| \sup_{|t| \leq 1} |T\phi_s|)^\theta \). Inequality (2.5) holds because of (2.4), (2.6) follows from the definition of distance induced by a Riemannian metric (note that \( s \mapsto T\phi_s(v), 0 \leq s \leq t \), is a path in \( TM \) connecting \( v \) and \( T\phi_t(v) \)), and (2.7) follows from the first variation equation applied to the \( C^1 \) vector field \( Y \).

Now let \( v \in E^{ss} \) be arbitrary. Then, since \( b_v(t) \) is linear in \( v \), we have:

\[
|b_v(t)| = \|v\| \left| \int b_{\frac{\partial}{\partial t}} X(t) \right|
\leq \|v\| C |t|^\theta \left| \frac{\partial}{\partial t} \right|^\theta
\]

(2.9)

Clearly, (2.9) follows from (2.8). This completes the proof of the lemma. \( \square \)

Now set

\[ \sigma(x, t, s) = \int_0^s \lambda(\phi_s x, t) \, dr. \]

Observe that for every fixed \( s \), \( \sigma(x, t, s) \to 0 \) as \( t \to -\infty \), uniformly in \( x \).

The next step is to show that the flows \( f_t \) and \( \phi_s \) satisfy the following commutation relation:

**Lemma 2.2** For all \( x \in M \) and \( t, s \in \mathbb{R} \):

\[
f_t \phi_s(x) = \phi_{\sigma(x, t, s)} f_t(x).
\]

(2.10)

**Proof** Since the foliation \( W^{uu} \) is invariant with respect to \( f_t \), there is a function \( \mu : M \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) such that (2.10) holds with \( \mu \) instead of \( \sigma \). We have to show:

\( \mu = \sigma \).

Differentiate (2.10) (with \( \mu \) instead of \( \sigma \)) with respect to \( s \). We get:

\[
T_{f_t}(Y_{\phi_s x}) = \frac{\partial \mu}{\partial s}(x, t, s) Y_{f_t \phi_s(x)}.
\]
Solving for $\partial \mu / \partial s$ and using the definition of $\lambda$, gives $\frac{\partial \mu}{\partial s}(x, t, s) = \lambda(\phi_s x, t)$, which directly implies $\mu = \sigma$. \Box

The question of how $b_v(t)$ behaves under iteration of the flow $f_t$ on $v$ is answered by the following lemma.

**Lemma 2.3** For every $v \in E^{ss}_x$ and $s, t \in \mathbb{R}$,

$$b_v(s) = b_{Tf_t(v)}(\sigma(x, t, s)).$$

**Proof** We have:

$$b_v(s) = (\phi_s^* \alpha)(v)$$
$$= (\phi_s^* f_t^* \alpha)(v)$$
$$= [(f_t \phi_s)^* \alpha](v)$$
$$= \alpha(T(f_t \phi_s)(v))$$
$$= \alpha\left(T(\phi_{\sigma(x,t,s)} f_t)(v)\right)$$
$$= \alpha\left(T \phi_{\sigma(x,t,s)} T f_t(v) + \frac{\partial \sigma}{\partial x}(v) Y\right)$$
$$= \alpha\left(T \phi_{\sigma(x,t,s)} T f_t(v)\right)$$
$$= b_{Tf_t(v)}(\sigma(x, t, s)),$$

as desired. Equality (2.11) follows from Lemma 2.2, (2.12) follows by the Chain Rule, and (2.13) holds because $\alpha(Y) = 0$. \Box

Set $\nu = \sup_{x \in M} |T_x f_1|_{E_x}$. Let $s \neq 0$, $|s| \leq 1$, and $v \in E^{ss}$ be arbitrary but fixed. We will show that $b_v(s) = 0$.

By Corollary 2.1.1 it follows that if $|s| \leq 1$, then

$$|\sigma(x, -t, s)| = \left| \int_0^s \lambda(\phi_{r} x, -t) \, dr \right|$$
$$\leq \left| \int_0^s e^{br} \lambda(x, -t) \, dr \right|$$
$$= e^b |s| \lambda(x, -t).$$

Since $\lambda(x, -t)$ tends to zero exponentially and $n > 3$, we can choose $\theta$ sufficiently close to 1 so that

$$\nu^{(n-3)\theta} \sigma(x, -t, s)^{\theta-1} \to 0,$$

(2.15)
as $t \to \infty$ uniformly in $x$. Let $C' = C e^t$. Then we have:

$$|b_v(s)| = |b_{Tf_{-t}(v)}(\sigma(x, t, s))|$$

$$\leq C |\sigma(x, -t, s)|^\theta \|Tf_{-t}(v)\|$$

$$= C |\sigma(x, -t, s)| \|Tf_{-t}(v)\| |\sigma(x, -t, s)|^{\theta-1}$$

$$\leq C' |s| \lambda(x, -t) \|Tf_{-t}(v)\| |\sigma(x, -t, s)|^{\theta-1}$$

$$= C' |s| \|Tf_{-t}(Y_x \land v)\| |\sigma(x, -t, s)|^{\theta-1}$$

$$\leq C' |s| |v| \cdot \nu^{(n-3)\mu} |\sigma(x, -t, s)|^{\theta-1}$$

$$\to 0,$$

as $t \to \infty$. Inequality (2.16) follows from Lemma 2.1 (clearly, we may assume $t$ is so large that $|\sigma(x, -t, s)| \leq 1$), (2.17) follows from (2.14), while (2.18) is a consequence of Lemma 1.1. The expression in (2.18) tends to zero by (2.15). Note that the hypothesis $n > 3$ is strongly used here.

Therefore, $b_v(s) = 0$ for every $v \in E^{ss}$ and $|s| \leq 1$. It is easy to verify that

$$b_v(t + s) = b_v(t) + b_{Z_v(t)}(s),$$

which immediately implies that $b_v(t) = 0$ for every $t \in \mathbb{R}$ and every $v \in E^{ss}$, i.e. $T\phi_t(v)$ has no component in the $X$-direction.

To complete the proof of Theorem 2.3, we need to show that the foliations $W^{ss}$ and $W^{uu}$ are jointly integrable. (This was discussed in Section 1.2.)

Let $U$ be an open subset of $M$ and let $L_0$ and $L_1$ be the plaques lying in $U$ of two leaves of the foliation $W^{cs}$. Choose $U$ so small that the projection $P : L_0 \to L_1$ along the leaves of $W^{uu}$ is well defined. There is a continuous function $\tau : L_0 \to \mathbb{R}$ such that for all $x \in L_0$,

$$P(x) = \phi_{\tau(x)}(x).$$

Since $W^{uu}$ is $C^1$, so are $\tau$ (by the Implicit Function Theorem) and $P$.

Now let $c : J \to L_0$ be a $C^1$ curve lying in a single strong stable leaf in $L_0$ ($J$ is some open interval in $\mathbb{R}$). Then by the Chain Rule:

$$\frac{d}{ds} [P(c(s))] = T\phi_{\tau(c(s))}(\dot{c}(s)) + d\tau(\dot{c}(s)) Y.$$
Therefore, \( TP(\dot{c}(s)) = \frac{d}{ds}[P(c(s))] \) has no component in the X-direction. Since \( TP(\dot{c}(s)) \) also belongs to \( E^{cs} \) for all \( s \in J \), it follows that, in fact, \( TP(\dot{c}(s)) \) belongs to \( E^{ss} \) for all \( s \). Since the path \( c \) in \( W^{ss} \) was arbitrary, we have proved

\[
P(W^{ss}(x) \cap L_0) \subset W^{ss}(P(x)) \cap L_1,
\]

for all \( x \in L_0 \). Therefore \( W^{ss} \) and \( W^{uu} \) are jointly integrable and the proof of Theorem 2.3 is complete. □

It is now natural to ask:

**Question** Given a volume preserving codimension one Anosov flow, is it possible to reparametrize it (or perturb it and get a flow topologically conjugate to the original one) so that for the new flow, \( E^{su} \) is of class Lip−?

This question will be discussed in some detail in the next chapter.
Chapter 3

Synchronization

In this chapter we show that it is possible to suitably reparametrize certain types of Anosov flows so that their properties become “synchronized”. We then give several applications of synchronization. The results are similar to some conclusions of Marcus [Ma1], [Ma2], Margulis [Mg] and Ghys [Gh1], [Gh2], but our methods differ from theirs.

3.1 On Reparametrizations of Anosov Flows

First we recall some basic facts.

Suppose \( \{ f_i \} \) is a reparametrization of an Anosov flow \( \{ f_t \} \) on \( M \) obtained by multiplying the generating vector field \( X \) by a \( C^1 \) function \( v: X' = vX \). Then \( f'_t \) is also an Anosov flow whose orbits coincide with the orbits of the original flow. In particular, \( f'_t \) admits a cross section if and only if \( f_t \) does. Furthermore, the center stable distributions of \( f'_t \) and \( f_t \) are identical, and the same is true for the center unstable ones. It is, in fact, possible to show that every vector in the strong stable distribution \( E^{ss'} \) of \( f'_t \) is of the form \( v + \xi(v)X \) for some \( v \in E^{ss} \), where \( \xi \) is some 1-form on \( E^{ss} \). (An analogous statement is true for the strong unstable distributions.) A word of caution is in place here: reparametrization does not preserve the strong distributions unless \( v \) is a constant function.

If \( f'_t \) is a reparametrization of a flow (not necessarily Anosov) \( f_t \) by a function \( v \),
then there exists a function \( \varrho : M \times \mathbb{R} \to \mathbb{R} \) such that
\[
f_t^x(x) = \varrho(x,t)(x)
\]
for all \( x \in M \) and \( t \in \mathbb{R} \). We state the following lemma and omit its straightforward proof.

**Lemma 3.1** (a) For all \( x \) and \( t \):
\[
\varrho(x, t) = \int_0^t v(f_s^x) \, ds.
\]

(b) \( \varrho(x, t + s) = \varrho(x, t) + \varrho(f_t^x, s) \quad (x \in M, \ t \in \mathbb{R}). \)

(c) For every \( x \in M \) and \( w \in T_x M \),
\[
T_x f_t^x(w) = T_x f_{\varrho(x,t)}(w) + \frac{\partial \varrho}{\partial x}(w) X.
\]

Let \( \{ f_t \} \) be a \( C^r \) (\( r \geq 2 \)) Anosov flow on \( M \). We make the following standing hypothesis for this section:

- \( \{ f_t \} \) is transitive (i.e. there is a dense orbit).
- \( E^{x,s} = E^{s,s} \oplus E^c \) is of class \( C^1 \).
- \( E^{uu} \) is orientable (but not necessarily 1-dimensional).

Let \( F \) be a \( C^\infty \) subbundle of \( TM \) complementary to \( E^{cs} \), i.e. \( \dim F = \dim E^{uu} = k \) and \( TM = E^{cs} \oplus F \), and choose a smooth Riemann structure \( \mathcal{R} \) on \( M \). Since \( F \) is orientable, it has a \( C^\infty \) volume \( k \)-form \( \omega_F \). Define a \( k \)-form \( \omega \) on the whole \( TM \) by extending \( \omega_F \) as follows:
\[
\text{Ker}(\omega) = E^{cs}, \quad \omega|_F = \omega_F.
\]

The first part of the definition just means that \( i_V \omega = 0 \), for every vector \( V \in E^{cs} \), where \( i_V \) denotes inner multiplication by \( V \). (Good references for elements of the calculus of differential forms are [Wa] or [GHL].)

Our next aim is to modify the existing Riemann structure \( \mathcal{R} \) on \( M \), so that with respect to the new structure \( \mathcal{R}_* \), \( \omega \) behaves nicely when pulled back by the Anosov flow.
Let $p : E^{uu} \to F$ be the bundle isomorphism given by orthogonal projection relative to $\mathcal{R}$. Define a new Riemann structure $\mathcal{R}_* \text{ on } M$ by declaring the following:

1. With respect to $\mathcal{R}_*$, $E^{uu}$ is orthogonal to $E^{cs}$.
2. $\mathcal{R}_*$ and $\mathcal{R}$ coincide on $E^{cs}$.
3. $p : (E^{uu}, \mathcal{R}_*) \to (F, \mathcal{R})$ is an isometry.

We call $\mathcal{R}_*$ the Riemann structure adapted to $\omega$ and $E^{uu}$. The important point in its definition is 3., as we shall see below.

Finally, let $\lambda(x, t)$ be the determinant of $T_x f_i |_{E^{uu}}$ relative to $\mathcal{R}_*$. Then we have the following result.

**Theorem 3.1** (a) For all $x \in M$ and $t \in \mathbb{R}$:

$$(f_i^* \omega)_x = \lambda(x, t) \omega_x.$$

(b) $\omega$ is $C^1$ and $d\omega = \eta \wedge \omega$ for some continuous 1-form $\eta$.

(c) Let $u = \eta(X)$; call it the “$u$-function” corresponding to $\omega$ and $X$. Then $u$ is $C^1$ and

$$u(f_i x) = \frac{d}{ds} \log \lambda(x, s).$$

**Proof** (a) First note the following. If $C$ is an $\mathcal{R}_*$-unit cube in $E^{uu}$, then, by construction, $p(C)$ is an $\mathcal{R}$-unit cube in $F$. Furthermore, if $C = Y_1 \ldots Y_k$, then $p(Y_j) - Y_j \in E^{cs}$, so $\omega(C) = \omega(p(C))$. Since $\omega|_F$ was an $\mathcal{R}$-volume form for $F$, we have:

$$\omega(C) = \omega(p(C)) \equiv 1.$$

Now let $U \subset M$ be a small open set over which $E^{uu}$ is trivial and let $x \mapsto C_x$ be a continuous family of $\mathcal{R}_*$-unit cubes in $E^{uu}$, for $x \in U$. Then for small $t$, $f_i * (C) = \lambda(x, t) C_{f_i x}$. Therefore,

$$(f_i^* \omega)_x (C_x) = \omega(f_i * (C_x))$$

$$= \lambda(x, t) \omega_{f_i x} (C_{f_i x})$$

$$= \lambda(x, t) \omega (C_x),$$
hence, \((f_t^*\omega)_x = \lambda(x, t) \omega_x\), for small \(t\). But by the cocycle property of \(\lambda\), this identity extends over all \((x, t) \in M \times \mathbb{R}\).

(b) By our standing assumption in this section that \(E^{cs}\) is \(C^1\), and since \(F\) is \(C^\infty\), it follows that \(\omega\) is \(C^1\). The second part of (b) is just a restatement of Frobenius theorem for the integrable distribution \(E^{cs}\).

(c) To show that \(u\) is \(C^1\), let \(Z_1, \ldots, Z_k\) be a \(C^\infty\) \(\mathcal{R}\)-orthonormal frame in \(F|_U\), which trivializes \(F\) over some small open set \(U\), and such that \(\omega(Z_1, \ldots, Z_k) \equiv 1\). Then we have:

\[
\begin{align*}
  u &= \eta(X) \\
  &= \eta(X) \omega(Z_1, \ldots, Z_k) \\
  &= (\eta \wedge \omega)(X, Z_1, \ldots, Z_k) \\
  &= d\omega(X, Z_1, \ldots, Z_k) \\
  &= X(\omega(Z_1, \ldots, Z_k)) + \sum_{i=1}^k (-1)^i Z_i(\omega(X, Z_1, \ldots, \hat{Z}_i, \ldots, Z_k)) + \\
  &\quad + \sum_{i<j} (-1)^{i+j} \omega([Z_i, Z_j], Z_0, \ldots, \hat{Z}_i, \ldots, \hat{Z}_j, \ldots, Z_k) \\
  &= \sum_{i<j} (-1)^{i+j} \omega([Z_i, Z_j], Z_0, \ldots, \hat{Z}_i, \ldots, \hat{Z}_j, \ldots, Z_k),
\end{align*}
\]

where \(Z_0 = X\) and the hat denotes omission. The expression in (3.4) is of class \(C^1\) because both \(\omega\) and \([Z_i, Z_j]\) are \(C^1\), for all \(i, j\). Identity (3.1) holds because \(i_X \omega = 0\), (3.2) follows from Frobenius theorem \((d\omega = \eta \wedge \omega)\), and (3.3) can be found in [Wa] (Proposition 2.25(f), p.70). Finally, (3.4) holds because \(\omega(Z_1, \ldots, Z_k)\) is identically equal to 1 (by definition), so its \(X\)-derivative is 0, and \(Z_i(\omega(X, Z_1, \ldots, \hat{Z}_i, \ldots, Z_k)) = Z_i(0) = 0\).

To show the second part of (c), consider (3.2) and recall that the Lie derivative (with respect to \(X\)), \(L_X\), can be written as \(L_X = di_X + i_X d\). Since \(di_X\omega = d0 = 0\), and (3.2) is equal to \((i_X d\omega)(Z_1, \ldots, Z_k)\), we have:

\[
  u \equiv (L_X \omega)(Z_1, \ldots, Z_k)
\]
\[ \frac{d}{dt} \left( f_t^* \omega \right)(Z_1, \ldots, Z_k) \]
\[ = \frac{d}{dt} \lambda(x, t). \]

Identity (3.5) is an alternative definition of Lie derivative of a differential form. The proof can now be completed in exactly the same way as in part (b) of Theorem 2.1. \( \square \)

Observe that \( u \) measures the rate of expansion of the flow along \( E^{uu} \). If \( u(x) < 0 \) for some \( x \), then in a neighborhood of \( x \) in \( M \), \( T f_t |_{E^{uu}} \) might not really be expanding for small \( t \). To avoid this unnatural and temporary behavior, we show that it is possible to modify the Riemann structure \( \mathcal{R}_* \) (and therefore modify \( \lambda \)) in a continuous fashion to obtain \( u > 0 \). Thus, after this modification, \( f_t \) becomes “immediately expanding” on \( E^{uu} \).

**Lemma 3.2** Let \( \{f_t\} \) be as above. Then there exists a continuous Riemann structure on \( M \) with respect to which \( u > 0 \).

**Proof** Recall that \( \lambda(x, t) \geq Ke^{ct} \), for all \( x \in M \), \( t > 0 \) and some \( K, c > 0 \). If \( K \geq 1 \), then by Theorem 3.1 (c) (see also Theorem 2.1 (c) for a similar calculation),

\[ \frac{1}{\tau} \int_{0}^{\tau} u(f_t x) \, dt \geq \frac{1}{\tau} \log K + c \geq c > 0, \]

for every \( x \in M \) and \( \tau > 0 \), so by letting \( \tau \rightarrow 0^+ \), we obtain \( u(x) > 0 \), and we are done.

If \( K < 1 \), choose \( \tau_0 > 0 \) so that \( c_0 = (\log K)/\tau_0 + c > 0 \). Again by Theorem 3.1 (c), if a periodic point \( x \) has period \( \tau \geq \tau_0 \), then

\[ \frac{1}{\tau} \int_{0}^{\tau} u(f_t x) \, dt \geq c_0 > 0. \]

Let \( P \) be the union of all periodic points of \( \{f_t\} \) whose period is longer than \( \tau_0 \). Then there are only finitely many periodic orbits of \( \{f_t\} \) which are not in \( P \) (for details see [PM], p.100), so by transitivity of the flow, \( P \) dense in \( M \). A standard argument (see, for instance, [Gh3], Lemma 2.4) shows that every \( f_t \)-invariant Borel
probability measure on $M$ can be approximated by convex combinations of invariant probabilities concentrated on periodic orbits in $P$, so the previous inequality implies 

$$\int_M u \, d\nu \geq c_0 > 0,$$

for every invariant Borel probability measure $\nu$ on $M$. Approximate $u$ by a $C^\infty$ function $w$ such that $\delta = \sup_M |u - w| < c_0/4$. Let $u_0 = w - \delta$. Then 

$$u_0 = (w - u - \delta) + u \leq u,$$

and for any invariant probability measure $\nu$ on $M$:

$$\int_M u_0 \, d\nu = \int_M (u_0 - u) \, d\nu + \int_M u \, d\nu \geq -2\delta + c_0 \geq \frac{1}{2} c_0.$$

The following “sublemma” due to Ghys [Gh3] makes use of this property of $u_0$.

**Sublemma**  There exists a $C^\infty$ function $v : M \to \mathbb{R}$ such that $X(v) < u_0$.

**Sketch of proof**  We only give the main steps, closely following the proof of Lemma 2.5 in [Gh3] where the reader can find all the details. The proof actually works for any smooth flow, independently of the Anosov property.

Let $u_0$ be as above and define

$$u_T(x) = \frac{1}{T} \int_0^T u_0(f_t(x)) \, dt.$$ 

It can be shown that for $T > 0$ large enough, $u_T < 0$. So to prove the lemma it is enough to construct a function $v_T$ such that

$$u + X(v_T) = u_T.$$ 

Consider the probability measures $\delta_0$ and $\gamma_T$ on $\mathbb{R}$, where $\delta_0$ is the Dirac mass at 0 and $\gamma_T$ is uniformly distributed on $[0,T]$. Then $\gamma_T - \delta_0$ is the generalized derivative of the function $h$ defined by:

$$h(t) = \begin{cases} 
\frac{t}{T} - 1, & \text{if } 0 \leq t \leq T, \\
0, & \text{otherwise}.
\end{cases}$$
That is, for every smooth $\phi : \mathbb{R} \rightarrow \mathbb{R}$,

$$
\int_0^T (1 - \frac{t}{T}) \phi'(t) \, dt = \frac{1}{T} \int_0^T \phi(t) \, dt - \phi(0).
$$

Thus for $v_T$ we can choose:

$$
v_T(x) = \int_0^T (1 - \frac{t}{T}) u_0(f_t x) \, dt.
$$

This completes the proof of the sublemma.

Continuing with the proof of Lemma 3.2, consider the 1-form

$$
\bar{\omega} = e^{-v} \omega.
$$

It is of class $C^1$ and defines the same plane field as $\omega$, namely $E^{cs}$. It is easy to see that

$$
d\bar{\omega} = (\eta - dv) \wedge \bar{\omega}.
$$

Furthermore, $(\eta - dv)(X) = u - X(v) \geq u_0 - X(v) > 0$. Let $\bar{\eta} = \eta - dv$ and $\bar{u} = \bar{\eta}(X)$. Note that $\bar{\eta}$ is the $u$-function corresponding to $\bar{\omega}$ and $X$. We have: $\bar{u}(x) > 0$ for all $x \in M$ and $\bar{\eta}$ satisfies part (c) of Theorem 3.1 if $\lambda(x, t)$ is taken relative to the Riemann structure adapted to $\bar{\omega}$ and $E^{uu}$. This completes the proof of Lemma 3.2. \(\square\)

Thus, without loss of generality we may assume that, with respect to some continuous Riemann structure on $M$, the $u$-function corresponding to $\omega$ and $X$ satisfies: $u > 0$.

Since $u > 0$, the vector field

$$
X' = \frac{1}{u} X
$$

is well defined. Let $\{f^i_t\}$ denote its flow. Then $f^i_t$ is an Anosov flow and its center distributions coincide with the corresponding center distributions of $f_t$. Clearly, $f^i_t$ is also transitive. Let $E^{uu'}$ be the strong unstable distribution of $f^i_t$. As remarked above, there is a 1-form $\xi$ such that every vector $w$ in $E^{uu'}$ can be written in the form $w = v + \xi(v) X$ for some $v \in E^{uu}$. Therefore, $E^{uu'}$ is an orientable bundle. Having verified the hypothesis at the beginning of the section, the synchronization procedure
used above will now be applied to the flow \( \{ f_t \} \). Observe that \( f_t \) is only of class \( C^1 \); however, this is not a problem simply because we do not need “\( C^r \)”-ness” of the flow any more (we originally needed it with \( r \geq 2 \) because the Invariant Section Theorem of [HPS] requires that the bundle map be at least as differentiable as the invariant distribution obtained by it). So we proceed in the following manner.

First we perform the “adaptation” (as described above) of the Riemann structure to \( \omega \) and \( E^u_{\omega'} \), and let \( \lambda'(x, t) \) be the determinant of \( T_x f_t \) on \( E^u_{\omega'} \) with respect to the adapted metric. Since the center stable distribution of \( f_t \) coincides with the one for the original flow, we can “recycle” \( \omega \) (which is defined by \( E^c \omega \)); hence we can also “recycle” \( \eta \). Set \( u' = \eta(X') \). This is the \( u \)-function corresponding to \( \omega \) and \( X' \). Therefore, the formula in Theorem 3.1(c) remains valid in this new setting; namely,

\[
\frac{d}{ds} \bigg|_{t} \log \lambda'(x, s) = \eta(X') = u(f_t x).
\]

However,

\[
u' = \eta \left( \frac{1}{u} X \right) = 1,
\]

and hence

\[
\lambda'(x, t) \equiv e^t.
\]

Thus we have proved the following result:

**Theorem 3.2** Let \( \{ f_t \} \) be a transitive \( C^r \) \( (r \geq 2) \) Anosov flow on a compact manifold \( M \), such that its center stable distribution is of class \( C^1 \) and its strong unstable distribution is orientable. Then there exists a continuous Riemann structure on \( M \) and a \( C^1 \) reparametrization \( \{ f'_t \} \) of \( \{ f_t \} \) such that the determinant of \( f'_t \) on its strong unstable distribution is identically equal to \( e^t \).

Moreover, if the center stable bundle of the original flow is of class \( C^s \) for some \( s \geq 1 \), then the new flow is of class \( C^{\min(r, s)} \); noninteger values of \( r \) and \( s \) are allowed.

We will call \( \{ f'_t \} \) the *synchronization* of the flow \( \{ f_t \} \).

We suppose that the reader is now curious to see what are some possible advantages of the synchronized flow over the original flow, so we refer to the next two sections where we focus on codimension one flows for which these advantages are most visible.
3.2 Applications of Synchronization

We will say that a Lie group $G$ acts \emph{locally freely} on a manifold $M$ if the isotropy group (or stabilizer) of the action at every point is a discrete subgrou of $G$. In that case, the orbits of the action form a foliation of $M$ of dimension $\dim G$.

Denote by $G$ the Lie group of orientation preserving affine transformations of the real line. Then we have the following

\textbf{Theorem 3.3} \textit{If a compact manifold $M$ admits a $C^2$ transitive codimension one Anosov flow whose strong unstable bundle is orientable, then $G$ acts locally freely on $M$. This action (i.e. the map $G \times M \to M$) is of class $C^1$.}

\textbf{Proof} Let $\mathfrak{g}$ be the Lie algebra of $G$. It is well known (see, for instance, [Wa]) that $\mathfrak{g}$ has two generators, $A$ and $B$, which satisfy the relation

$$[A, B] = -B.$$ 

Let $X$ be the Anosov field of the synchronized flow $\{f_t\}$ and $Y$ a section of its strong unstable bundle such that $Tf_t(Y) = e^t Y$. (This can be done by Theorem 3.2.) Then, clearly, $[X,Y] = -Y$, which implies that $G$ acts on $M$. The orbits of this action are the leaves of the center stable foliation of the synchronized flow. Let $x \in M$ be arbitrary, and denote by $S_x$ the stabilizer of the action at $x$, that is, the set of all elements of $G$ which have $x$ as a fixed point. If $x$ is a periodic point of the Anosov flow, then $S_x$ is free cyclic, otherwise $S_x$ is trivial. Therefore, the action of $G$ on $M$ is locally free. It is not difficult to see that it is also of class $C^1$. $\square$

It is easy to see that the synchronization of a volume preserving flow is volume preserving: if the flow of $X$ preserves a volume form $\Omega$, and $X' = \frac{1}{u}X$ is the synchronized vector field, then the flow of $X'$ preserves the volume form $\Omega' = u \Omega$:

\begin{align*}
L_{X'} \Omega' &= (d i_{X'} + i_{X'} d) \Omega' \\
&= d i_{X'} \Omega' \\
&= d i_X (\frac{1}{u} \Omega) \\
&= L_X \Omega \\
&= 0.
\end{align*}
For simplicity of notation denote the synchronized flow of a volume preserving codimension one Anosov flow by \( \{f_t\} \), the vector field tangent to it by \( X \) and assume it preserves a \( C^1 \) volume form \( \Omega \). Let \( Y \) be a continuous unit vector field in the strong unstable bundle of \( \{f_t\} \) which is the projection to \( E^{uu} \) of some \( C^\infty \) vector field \( Z \) everywhere transverse to \( E^{cs} \) (cf. the previous section). Define 1-forms \( \alpha \) and \( \omega \), and an \( (n-2) \)-form \( \theta \) by:

- \( \ker(\alpha) = E^{su}, \alpha(X) = 1; \)
- \( \ker(\omega) = E^{cs}, \omega(Y) = 1; \)
- \( \theta = i_X i_Y \Omega. \)

Then \( \omega \) is of class \( C^1 \), while \( \alpha \) and \( \theta \) are only continuous. However, the nullspace of \( \theta \), i.e. \( E^{cu} \), is an integrable distribution so by Hartman’s version of Frobenius theorem \( d\theta \) exists in the Stokes sense. Furthermore we have:

**Theorem 3.4** (a) \( d\omega = \alpha \wedge \omega. \)

(b) \( d\theta = -\alpha \wedge \theta. \)

(c) \( \alpha \) is closed on the leaves of \( W^{cs} \) and \( W^{cu}. \)

**Proof** (a) By Frobenius theorem there is a continuous form \( \eta \) such that \( d\omega = \eta \wedge \omega. \) Since \( \det T_{f_t} \) on \( E^{uu} \) is identically \( e^t \), we have \( f_t^* \omega = e^t \omega \) for all \( t \). It follows \( f_t^* (d\omega) = df_t^* \omega = d(e^t \omega) = e^t d\omega, \) hence

\[
(f_t^* \eta) \wedge \omega = e^{-t} f_t^* (\eta \wedge \omega) = e^{-t} f_t^* (d\omega) = e^{-t} e^t \eta \wedge \omega = \eta \wedge \omega.
\]

Elementary calculus of differential forms implies that there exists a continuous function \( k : \mathbb{R} \times M \to \mathbb{R} \) such that

\[
f_t^* \eta - \eta = k(t,x) \omega. \tag{3.6}
\]
Evaluate both sides of (3.6) at an arbitrary vector \( v \in E^{ss} \). Since \( \omega(v) = 0 \), we obtain

\[
|\eta(v)| = |(f_t^* \eta)(v)|
\]
\[
= |\eta(Tf_t(v))|
\]
\[
\leq |\eta|_{\infty} \|Tf_t(v)\|
\]
\[
\to 0,
\]
as \( t \to \infty \). Since we know from before that \( \eta(X) = 1 \) (see Section 3.1), it follows that the restrictions of \( \alpha \) and \( \eta \) to the distribution \( E^{cs} \) are identical. Thus, there is a function \( h : M \to \mathbb{R} \) such that \( \eta = \alpha + h \omega \), which implies

\[
\eta \wedge \omega = \alpha \wedge \omega,
\]
as desired. Note that the proof doesn’t use volume preservation.

(b) By Frobenius theorem there exists a continuous 1-form \( \beta \) such that \( d\theta = \beta \wedge \theta \). We will show that the restrictions of \( \beta \) and \( -\alpha \) to \( E^{cu} \) coincide.

Observe that the determinant of \( Tf_t \) on \( E^{ss} \) is identically \( e^{-t} \). So just as we showed that \( f^*_t \omega = e^t \omega \), we can in exactly the same fashion show that

\[
f_t^* \theta = \det Tf_t|_{E^{cu}} = e^{-t} \theta,
\]
i.e. \( L_X \theta = -\theta \). Since \( L_X = di_X + i_X d \) and \( i_X \theta = 0 \), we have:

\[
-\theta = i_X d \theta
\]
\[
= i_X (\beta \wedge \theta)
\]
\[
= \beta(X) \theta.
\]
Thus \( \beta(X) = -1 \). Following the procedure in part (a), it is easy to show that \( \beta(Y) = 0 \). Hence \(-\beta\) and \( \alpha \) are identical as forms on the leaves of \( W^{cu} \) and therefore \( d\theta = -\alpha \wedge \theta \), as desired.

(c) Part (a) implies (see, for instance [HH]) that the integral of \( \alpha \) over any loop contained in a leaf of \( W^{cs} \) equals the logarithm of the linear part of the holonomy of the foliation \( W^{cs} \). So if \( D \) is a disk contained in a leaf of \( W^{cs} \), then \( \int_{\partial D} \alpha = 0 \), since
trivial loops (such as $\partial D$) carry no holonomy. Thus the Stokes differential of $\alpha$ on each leaf of $W^{cs}$ is zero.

Similarly, the integral of $-\alpha$ over a loop contained in a leaf of $W^{cu}$ is equal to the logarithm of the absolute value of the determinant of the holonomy of the foliation $W^{cu}$. The same argument as above then shows that $-\alpha$ is closed in the Stokes sense on every leaf of $W^{cu}$. $\square$

### 3.3 Invariant Forms and Godbillon-Vey Class of the Center Stable Foliation

In this section we consider bounded forms invariant with respect to a codimension one Anosov flow. We already saw in Chapter 1 that such forms of degree two must vanish almost everywhere; here we extend that result and apply it to a discussion of the Godbillon-Vey class of the center stable foliation.

**Theorem 3.5** Every bounded $k$-form on a compact manifold $M$ of dimension $n > 3$ which is invariant with respect to some codimension one Anosov flow on $M$ vanishes if $2 \leq k \leq n - 2$.

**Proof** Let $\beta$ be bounded $f_t$-invariant form of degree $k$ on $M$, with $2 \leq k \leq n - 2$, where $f_t$ is some codimension one Anosov flow on $M$. Since the proof in the case $k = 2$ will be obvious from what follows, we will assume $k > 2$.

Choose a continuous Riemannian metric on $M$ relative to which the bundles $E^c$, $E^{ss}$ and $E^{uu}$ orthogonal. As before, let $X$ be the Anosov vector field and let $Y$ be a continuous section of $E^{uu}$. Without loss we may assume $X$ and $Y$ have unit length. Let $x \in M$ be an arbitrary point and at $x$, choose an arbitrary orthonormal basis of $E^{ss}$: $v_1, \ldots, v_{n-2}$. It suffices to show that

$$\beta(X, Y, v_1, \ldots, v_{k-2}) = 0$$
and

$$\beta(Y, v_1, \ldots, v_{k-1}) = 0.$$ 

Let \( C = X \land Y \land v_1 \land \ldots \land v_{n-2} \) be the corresponding unit cube in \( T_x M \). Denote by \( \Delta(x, t) \) the determinant of \( T_x f_t \). Thus \( \Delta(x, t) = \| f_{ts}(C) \| \). As shown by Plante in [Pl1], Egoroff’s theorem implies that the set of points \( x \) for which \( \Delta(x, t) \) tends to \( \infty \) as \( |t| \to \infty \) has Lebesgue measure zero. Therefore for a.e. \( x \in M \), there is a sequence \( (t_i) \) converging to \( \infty \) such that \( \Delta(x, t_i) \) stays bounded as \( i \to \infty \). Call such a sequence \( (t_i) \) a good sequence and the corresponding points (for which there is a good sequence), good points. Consider the following two cases:

**Case 1** Let \( \nu > 1 \) be such that \( \| T f_{-t}(v) \| \geq \nu \| v \| \), for large \( t \) and \( v \in E^{ss} \). Set \( B = |\beta|_{\infty} \). If \( x \) is a good point, then:

\[
|\beta(X, Y, v_1, \ldots, v_{k-2})| = |(f_{-t}^* \beta)(X, Y, v_1, \ldots, v_{k-2})| = |\beta(f_{-t} (X \land Y \land v_1 \land \ldots \land v_{k-2}))| \\
\leq B \| f_{-t} (X \land Y \land v_1 \land \ldots \land v_{k-2}) \| \frac{\| f_{-ts}(C) \|}{\| f_{-ts}(C) \|} \\
\leq B \Delta(x, t) \nu^{-(n-k)t} \\
\to 0,
\]

where \( t \to \infty \) along a good sequence. Note that \( n - k \geq 2 \).

**Case 2** Similarly, if \( x \) is a good point, we have:

\[
|\beta(Y, v_1, \ldots, v_{k-1})| = |(f_{-t}^* \beta)(Y, v_1, \ldots, v_{k-1})| = |\beta(f_{-t} (Y \land v_1 \land \ldots \land v_{k-1}))| \\
\leq B \| f_{-t} (Y \land v \land v_{k-1} \land v_{n-2}) \| \frac{\| f_{-ts}(C) \|}{\| f_{-ts}(C) \|} \\
\leq B \Delta(x, t) \nu^{-(n-k-1)t} \\
\to 0,
\]

as \( t \to \infty \) along a good sequence. Observe that \( n - k - 1 \geq 1 \).
Since the set of good points has full measure in $M$, the proof is now complete. □

Let us apply the previous theorem to a calculation of the Godbillon-Vey class of the center stable foliation of a codimension one Anosov flow. Recall that for a $C^2$ codimension one foliation $\mathcal{F}$ whose tangent bundle is the nullspace of a $C^2$ 1-form $\omega$, the Godbillon-Vey class of $\mathcal{F}$, $GV(\mathcal{F})$, is a class in the 3rd de Rham cohomology space $H^3(M, \mathbb{R})$ of the underlying manifold $M$, defined in the following way. By Frobenius theorem there exists a $C^1$ form $\eta$ such that $d\omega = \eta \wedge \omega$. It can be shown (see [To]) that the de Rham cohomology class of the 3-form $\eta \wedge d\eta$ does not depend on the choice of $\eta$, as well as on the choice of $\omega$. So we can define

$$GV(\mathcal{F}) = \{ \eta \wedge d\eta \},$$

where the curly braces signify de Rham cohomology class.

**Remark 3.1** The definition of the Godbillon-Vey class was extended by Hurder and Katok in [HK] to codimension one foliations of class $C^{1+\epsilon}$ on 3-manifolds where $\epsilon > 1/2$. It follows from our discussion of Lipschitz forms in Chapter 1 that the Godbillon-Vey class is defined when the codimension one foliation is $C^{1+\text{Lip}}$. However, despite the general feeling of experts that the Godbillon-Vey class is definable in higher dimensions ($> 3$) for codimension one foliations of class $C^{1+\epsilon}$ with $\epsilon > 1/2$, I am still not aware of any written proof of that.

Now consider the center stable foliation of a codimension one Anosov flow in dimension $> 4$ and assume that it is possible to define its Godbillon-Vey class, $GV$. Consider the synchronization $f_t$ of the original flow and the Godbillon-Vey class of its center stable foliation which, clearly, coincides with $GV$. Theorem 3.4 suggests that, since $d\omega = \alpha \wedge \omega$, and $GV = \{ \alpha \wedge d\alpha \}$ (where it remains to make sense out of $d\alpha$ in case when $\alpha$ is just Hölder), there is a representative of $GV$ which is invariant with respect to $f_t$. Theorem 3.5 then implies that $GV = 0$. Because of this we make the following
**Conjecture**  If the Godbillon-Vey class of the center stable foliation of a codimension one Anosov flow in dimension $> 4$ is definable, then it automatically vanishes.

The following theorem gives a positive answer to the conjecture provided that $W^{cs}$ is differentiable enough. Part (a) of the theorem appears in the paper [Gh3] with incomplete proof. (Namely, in [Gh3] the following result from [Ve] is used, the proof of which is incorrect: The lift of the center stable foliation of any codimension one Anosov flow to the universal covering space is given by a $C^1$ submersion. Etienne Ghys has informed me that he has an alternative unpublished proof of his result. See also Chapter 4 for further discussion of this question.)

**Theorem 3.6** Let $\{f_t\}$ be a $C^2$ volume preserving codimension one Anosov flow on a compact Riemannian manifold $M$ of dimension $n > 3$. If the center stable distribution $E^{cs}$ of the flow is of class $C^{1 + \text{Lip} -}$, then:

(a) The flow admits a global cross section.

(b) The Godbillon-Vey class of $W^{cs}$ is zero.

**Proof** (a) First note that the main obstacle in this proof is that the strong unstable bundle of the synchronization of $\{f_t\}$ does not have to be of class $C^1$, despite volume preservation. The reason is simple: the synchronized flow is not necessarily of class $C^2$, so the $C^1$ section theorem does not apply. However, we claim that the strong unstable bundle of the synchronized flow is $\text{Lip} -$.

To see this, assume for simplicity of notation that the original flow has been synchronized; call it $\{f_t\}$ and let the corresponding splitting be $TM = E^c \oplus E^{ss} \oplus E^{uu}$. Since the $u$-function used in reparametrization is as smooth as $W^{cs}$ (see Theorem 3.2), it follows that $T_{f_{-1}}$ is $\text{Lip} -$. Let the Lipschitz constant of $f_{-1}$ be $\mu$. Clearly, $\mu = \|T_{f_{-1}}|_{E^u}\| > 1$. Note that the norm of $T_{f_{-1}}$ on $E^{uu}$ is $1/e$, because the $E^{uu}$-determinant has been synchronized. Since $\dim M > 3$, we have that $\mu/e < 1$, and, in particular, $\mu^\theta/e < 1$, for every $\theta < 1$. The Hölder section theorem (see [Sh], Theorem 5.18 (c)) now implies that $E^{uu}$ is $\theta$-Hölder for all $\theta < 1$, as desired.

By Theorem 3.4, we may assume

$$d\omega = \alpha \wedge \omega, \quad (3.7)$$
where ω and α have the same meaning as before. Note that (3.7) alone does not yet imply that α is of class Lip−, which is what we would like to show. However, since ω is $C^{1+}\text{Lip}^-$, there exists a Lip− form η such that $dω$ is also equal to $η \land ω$. Elementary calculus of differential forms implies that $η - α$ is a multiple of $ω$, i.e. there exists a continuous function $h : M \to \mathbb{R}$ such that

$$η = α + h \omega.$$  \hspace{1cm} (3.8)

Evaluate both sides of (3.8) at $Y$, a nonsingular Lip− section of $E^{uu}$ such that $ω(Y) = 1$. Then since $α(Y) = 0$, we have:

$$η(Y) = h.$$

Therefore, $h$ is of class Lip−, hence so is $α$. By Theorem 2.3, it follows that \{fi\} admits a global cross section.

(b) By part (a) and Theorem 2.3 we actually have that $W^{ss}$ and $W^{uu}$ are jointly integrable. This, by [P11], implies that $E^{ss}$ is integrable. Therefore, by [Ha], $dα$ exists in the Stokes sense, and $α \land dα = 0$. It is easily seen that this implies $dα = 0$. Thus

$$GV(W^{cs}) = \{α \land dα\} = 0,$$

as desired. □

**Remark** Hurder and Katok showed that if $\dim M = 3$, then the center stable foliation is, indeed, of differentiability class required in Theorem 3.6 (in fact, they showed even more; see [HK]). It is an open question under which conditions the same statement is true in higher dimensions.

Note that a slight modification of the proof of Theorem 3.6 gives us the following simple lemma on differential forms:

**Lemma 3.3** Suppose $α, η$ and $ω$ are 1-forms on a manifold $M$ and $Y$ is a $C^1$ vector field on $M$. If $η$ and $ω$ are of class $C^1$, $ω(Y) > 0$, $α(Y) = 0$ and $α \land ω = η \land ω$, then $α$ is also of class $C^1$. 
Corollary 3.6.1 If \( \{f_i\} \) is a \( C^2 \) volume preserving codimension one Anosov flow on a compact Riemannian manifold \( M \) of dimension \( n > 3 \) and its center stable foliation \( W^{cs} \) is of class \( C^2 \), then both the strong stable and strong unstable bundle of the synchronization of \( \{f_i\} \) are of class \( C^1 \).

Proof It follows from Lemma 3.3 that the \( E^{su} \)-bundle (call it \( F \)) of the synchronized flow \( \{f_i\} \) as well as its strong unstable bundle (call it \( F^{uu} \)) are of class \( C^1 \). Let us show that the strong stable bundle, \( F^{ss} \) of the synchronized flow is also \( C^1 \). We will use the \( C^1 \) section theorem of [HP]. More specifically, we closely follow the proof of Theorem 6.3 of [HP].

Approximate in the \( C^0 \) sense \( F^{ss} \) by a \( C^1 \) subbundle \( F^0 \) of \( F \). For each \( x \in M \) let \( L_x \) be the space of linear maps \( F^0 \to F^{uu} \) with norm \( \leq 1 \). We seek \( F_x^{ss} \) as the graph of an element of \( L_x \), that is, we are looking for a section of the bundle \( L \to M \) invariant relative to \( Th \), where \( h = f_{-1} \).

Let \( \Gamma : L \to L \) be the graph transform induced by \( Th : F^0 \oplus F^{uu} \to F^0 \oplus F^{uu} \). If the matrix of \( Th \) relative to the splitting \( F^0 \oplus F^{uu} \) is

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix},
\]

then \( \Gamma \) is defined by the formula:

\[
\Gamma(\sigma) = (C + D\sigma) \cdot (A + B\sigma)^{-1},
\]

where \( \sigma \) is section of \( L \to M \). Let \( \epsilon > 0 \) be as small as we want. Then we can choose \( F^0 \) so close to \( F^{ss} \) that the Lipschitz constant of \( \Gamma \), \( L(\Gamma) \), can be estimated as follows:

\[
k := L(\Gamma) \leq \|D\|\|A^{-1}\| + \epsilon \\
\leq \mu \nu + \epsilon,
\]

where \( \mu = \|T f_{-1}|_{F^{uu}}\| < 1 \) and \( \nu = \|T f_1|_{F^{uu}}\| < 1 \). On the other hand, for the base map \( h \) we have the following estimate:

\[
\ell := L(h^{-1}) = L(f_1) \leq \|T f_1|_{F^{uu}}\|.
\]
If $\epsilon$ is sufficiently small, then $k \ell < 1$. By the $C^1$ section theorem, it follows that $\Gamma$ has a unique invariant section which is of class $C^1$. Since that section must be $F^{ss}$, the proof of the corollary is complete. \ \Box

In [HK], Hurder and Katok asked whether a similar effect can be achieved by reparametrizing an Anosov flow with the same properties as above but in dimension three. This was answered positively by Ghys in [Gh2]. Our corollary gives a positive answer to a similar question in dimensions > 3.
Chapter 4

The Universal Covering Space

In this chapter we focus our attention on the lift of our codimension one Anosov flow to the universal covering space of the underlying manifold. It was shown by Palmeira [Pa] (as a consequence of a more general result) that every \( n \)-manifold admitting a codimension one Anosov flow is covered by Euclidean space \( \mathbb{R}^n \).

Let \( p : \mathbb{R}^n \to M \) be a \( C^\infty \) covering projection and let \( \bar{X}, \bar{f}, \bar{\omega}, \bar{E}^{ss}, \) etc. denote the lifts to \( \mathbb{R}^n \) by \( p \) of the corresponding objects on \( M \). Consider the following:

**Condition (L)** The lift \( \bar{W}^{cs} \) of the center stable foliation to \( \mathbb{R}^n \) is given by a \( C^1 \) submersion \( \mathbb{R}^n \to \mathbb{R} \).

This condition was studied in [Ve], and an incorrect proof of it was given. (More precisely, Proposition 3.4 of [Ve] is incorrect. This was pointed out to me by S. Fenley.) Since it will lead to some interesting consequences, we will take (L) as an additional assumption in the remaining part of this chapter. It is easy to see that (L) will be satisfied if, for instance, \( \bar{W}^{cs} \) admits a transversal intersecting all of its leaves. By Solodov [So] this is the case when the center of the fundamental group of \( M \) is free cyclic. Observe that if (L) holds, then we can assume that \( p \) is of class \( C^1 \) and \( \bar{W}^{cs} \) is the hyperplane foliation given by \( x_n = \text{constant} \).

So assume that (L) is satisfied. There exists a continuous positive function \( g : \)
\( \mathbb{R}^n \to \mathbb{R} \) such that
\[
\dot{\omega} = g \, dx_n. 
\] (4.1)

This is because \( \dot{\omega} \) and \( dx_n \) have the same nullspace at every point, namely \( \tilde{E}^{cs} \). (If \( g \) is negative, we can simply replace \( \omega \) by \( -\omega \).) As before, we choose a continuous 1-form \( \eta \) so that \( d\omega = \eta \wedge \omega \) and let \( u = \eta(X) \). Also, \( \lambda(x,t) = \det T_x f|_{E^u} \), with respect to some Riemann structure on \( M \).

Then we have the following result:

**Theorem 4.1**

(a) \( g \) has partial derivatives with respect to \( x_1, \ldots, x_{n-1} \) and they are continuous.

(b) There exists a continuous function \( b : \mathbb{R}^n \to \mathbb{R} \) such that
\[
\tilde{\eta} = d^{cs}(\log g) + b(x) \, dx_n. 
\]

Here \( d^{cs} \) denotes the leafwise \( \tilde{W}^{cs} \)-differential: \( d^{cs}(\log g) = \sum_{i=1}^{n-1} \partial \log g/\partial x_i \, dx_i \).

(c) We have: \( u(p(x)) = \tilde{X}_x(\log g) \) and
\[
\lambda(p(x), t) = g(\tilde{f}_t(x)) \quad \frac{g(x)}{g(x)},
\]

for every \( x \in \mathbb{R}^n \) and \( t \in \mathbb{R} \).

(d) For every deck transformation \( T \) and every \( x \in \mathbb{R}^n \),
\[
g(Tx) \, T_n'(x_n) = g(x).
\]

Here \( T_n \) denotes the \( n \)-th coordinate function of \( T \). Since \( T \) preserves \( \tilde{W}^{cs} \), \( T_n \) depends only on \( x_n \), so \( T_n'(x_n) = \frac{\partial T_n}{\partial x_n}(x_n) \).

**Proof**

(a), (b) We will use the notion of the generalized (or weak) differential of exterior differential forms.

**Definition 4.1**

We say that a locally integrable form \( \beta \) of degree \( r+1 \) is the generalized differential of a locally integrable \( r \)-form \( \alpha \) if for every \( C^\infty \) form \( \phi \) of degree \( n-r-1 \) with compact support the following holds:
\[
\int_{\mathbb{R}^n} \phi \wedge \beta = (-1)^{n-r} \int_{\mathbb{R}^n} d\phi \wedge \alpha.
\]
If $\alpha$ is of class $C^1$, then its ordinary differential is also its generalized differential. For details see [Rs], II.4.

Now consider the form $\tilde{\omega} = p^*\omega$. Since $p$ is only of class $C^1$, we cannot claim that $\tilde{\omega}$ is a $C^1$ form. However, using $p$ as a local change of variables, it is not difficult to see that $\tilde{\omega}$ is actually differentiable in the generalized sense. Indeed, let $\phi$ be a $C^\infty$ form of degree $n-2$ with compact support in $\mathbb{R}^n$. Without loss of generality we can assume the support of $\phi$ is contained in a small open set $U$ on which $p$ is 1-1 (otherwise use a partition of unity). Thus there exists an $(n-2)$-form $\psi$ on $M$ such that $\phi = (p|_U)^*\psi$ and $d\psi$ exists in the generalized sense. It follows

$$
\int_{\mathbb{R}^n} d\phi \wedge \tilde{\omega} = \int_U p^*(d\psi \wedge \omega) = \int_M d\psi \wedge \omega
= (-1)^{n-1} \int_M \psi \wedge d\omega
= (-1)^{n-1} \int_{\mathbb{R}^n} \phi \wedge p^*(d\omega).
$$

Therefore, $p^*(d\omega)$ is the generalized differential of $\tilde{\omega}$. Clearly, $p^*(d\omega) = p^*(\eta \wedge \omega) = \tilde{\eta} \wedge \tilde{\omega}$.

There exist continuous functions $a_i : \mathbb{R}^n \to \mathbb{R}$ $(1 \leq i \leq n)$ such that

$$
\tilde{\eta} = \sum_{i=1}^n a_i \, dx_i.
$$

Let $h : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary smooth function with compact support. Define $\phi = h \, dx_2 \wedge \ldots \wedge dx_{n-1}$. Then we have:

$$
\int_{\mathbb{R}^n} g \frac{\partial h}{\partial x_1} dx_1 \wedge \ldots \wedge dx_n = \int_{\mathbb{R}^n} d\phi \wedge \tilde{\omega}
= (-1)^{n-1} \int_{\mathbb{R}^n} \phi \wedge \tilde{\eta} \wedge \tilde{\omega}
= -\int_{\mathbb{R}^n} g a_1 \, h \, dx_1 \wedge \ldots \wedge dx_n.
$$

This shows that the generalized (i.e. Sobolev) derivative of $g$ with respect to $x_1$ is $g a_1$. Thus

$$
a_1 = \frac{\partial (\log g)}{\partial x_1},
$$
in the generalized sense. Similarly, we can prove analogous formulas for \( a_2, \ldots, a_{n-1} \). Since \( a_i \)'s are continuous hence locally bounded functions, \( g \) actually belongs to the Sobolev space \( W^{1,\infty}_{\text{loc}} \) on every hyperplane \( x_n = \text{constant} \). By the theory of Sobolev spaces and Rademacher’s theorem, \( g \) is differentiable in the ordinary sense with respect to the first \( n-1 \) variables and its weak partials equal its ordinary partials. It follows that \( g \) is continuously differentiable with respect to the first \( n-1 \) variables in the ordinary sense. This completes the proof of both (a) and (b).

(c) Follows straightforwardly from part (b) and Theorem 2.1.

(d) Let \( T \) be a deck transformation of the covering \( p : \mathbb{R}^n \to M \). Since \( T \) preserves the foliation \( \tilde{W}^{cs} \), it follows that its \( n^{th} \) component, \( T_n \), depends only on \( x_n \). Note that \( T^*\tilde{\omega} = \tilde{\omega} \) and \( T^*(dx_n) = T^*_n(x_n)\ dx_n \). Part (d) now easily follows by applying the pullback \( T^* \) to the relation \( \tilde{\omega} = g\ dx_n \).

This completes the proof of the theorem. \( \square \)

Therefore, \( g \) strictly increases along the orbits of the lifted flow provided that \( u > 0 \). If we synchronize the flow, we obtain \( u = 1 \), hence

\[ g(\tilde{f}_t x) \equiv e^t g(x), \]

where \( \tilde{f}_t \) denotes the lift of the synchronized flow to \( \mathbb{R}^n \). This easily implies that \( g \) is constant on the leaves of the foliation \( \tilde{W}^{ss} \) (for the synchronized flow). Thus we have proved the following result:

**Theorem 4.2** If a codimension one Anosov flow satisfies condition (L), then its lift to the universal covering space (that is, \( \mathbb{R}^n \)) admits a continuous global Lyapunov function which strictly increases along its orbits and is constant on the leaves of \( \tilde{W}^{ss} \) for the synchronized flow.

Let \( \mathcal{G} \) be the partition of \( \mathbb{R}^n \) by the level surfaces of \( g \). (Note that since \( g \) is not necessarily \( C^1 \), we cannot claim that the leaves of \( \mathcal{G} \) are smooth.) It is natural to ask whether \( \mathcal{G} \) is invariant with respect to the deck transformations \( T \). If so, by projecting
\( \mathcal{G} \) to \( M \), we would obtain a partition \( p(\mathcal{G}) \) of \( M \) which is invariant with respect to the synchronized flow \( f_t \). That would in turn imply that \( p(\mathcal{G}) \) is a continuous foliation tangent to \( E^{ss} \oplus E^{uu} \), hence the flow would be a suspension. Unfortunately, it does not seem easy to prove that \( \mathcal{G} \) is projectable to \( M \).

However, Barbot [Ba] showed that a sufficient condition for a codimension one Anosov flow (in any dimension) to have a cross section is that every leaf of \( \tilde{W}^{uu} \) intersects every leaf of \( \tilde{W}^{cs} \) exactly once. We state this condition in terms of the function \( g \). Namely, let (as before) \( Y \) be a nonvanishing continuous section of the bundle \( E^{uu} \) such that \( \omega(Y) = 1 \) and denote its lift to \( \mathbb{R}^n \) by \( \tilde{Y} \) and the lift of its flow by \( \{ \tilde{\phi}_t \} \). If \( x_n \) denotes the projection to the \( n^{th} \) coordinate in \( \mathbb{R}^n \), then:

\[
\frac{d}{dt} x_n(\tilde{\phi}_t x) = dx_n(\tilde{Y}_{\tilde{\phi}_t x}) = \frac{1}{g(\tilde{\phi}_t x)} \tilde{\omega}(\tilde{Y}) = \frac{1}{g(\tilde{\phi}_t x)}.
\]

Thus

\[
x_n(\tilde{\phi}_t x) = \int_0^t \frac{1}{g(\tilde{\phi}_s x)} \, ds. \tag{4.2}
\]

Since the condition of Barbot is satisfied if the integral on the right hand side of (4.2) diverges to \( +\infty \) as \( t \to +\infty \), and to \( -\infty \) as \( t \to -\infty \), we have the following:

**Lemma 4.1** A codimension one Anosov flow which satisfies condition (L) has a global cross section if for all \( x \in \mathbb{R}^n \) (with notation as above),

\[
\int_{0}^{\pm\infty} \frac{1}{g(\tilde{\phi}_s x)} \, ds = \pm\infty.
\]

It is not difficult to see that the divergence of this integral is equivalent to the complete integrability of the vector field \( Z = g \, \tilde{Y} \). In other words, that its flow lines through all points of \( \mathbb{R}^n \) are defined for all time. One could hope to prove this by finding the “true meaning” of \( g \) in terms of objects in the base manifold \( M \).
Final Remarks

We saw that the main difficulty we had to deal with was the lack of smoothness of strong Anosov distributions. One could get around this by proving a version of Frobenius theorem for continuous distributions with an integrability condition which would not use differentiability (which we have not been able to do). In fact, it seems plausible that it would suffice only to construct (possibly nonunique) integral manifolds for $E^{su}$ at every point; we have not attempted to do that in this paper.

To conclude, let us say that although the general case of Verjovsky’s conjecture remains to be an open problem, we hope that we will be able to give its proof in a not so distant future.
Bibliography


[To] Ph. Tondeur: *Foliations on Riemannian manifolds*, Universitext, Springer-Verlag, New York, 1988

