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Show your work
1. **(25 points)** Determine the form of a particular solution to the given ODE, but do not solve for the coefficients:

\[ y'' - 2y' + 2y = te^t \cos t + \cos 2t. \]

**Solution:** Since the roots of the associated auxiliary equation are 1±i, the general solution to the corresponding homogeneous equation is

\[ y_h = C_1 e^t \cos t + C_2 e^t \sin t. \]

To find a particular solution \( y_p \), we use the superposition principle, so that \( y_p = y_{p1} + y_{p2} \), where \( y_{p1} \) comes from \( te^t \cos t \) and \( y_{p2} \) from \( \cos 2t \).

An initial guess for \( y_{p1} \) is \( P_1(t)e^t \cos t + Q_1(t)e^t \sin t \), where \( P_1(t) \) and \( Q_1(t) \) are polynomials of degree one. However, since this guess overlaps with \( y_h \), we need to multiply it by \( t \), so

\[ y_{p1} = t[P_1(t)e^t \cos t + Q_1(t)e^t \sin t]. \]

On the other hand,

\[ y_{p2} = A \cos 2t + B \sin 2t. \]

Therefore,

\[
\begin{align*}
y_p &= y_{p1} + y_{p2} \\
&= t[P_1(t)e^t \cos t + Q_1(t)e^t \sin t] + A \cos 2t + B \sin 2t.
\end{align*}
\]
2. (25 points) Solve the initial value problem:

\[
y'' - 2y' + y = \frac{e^t}{t}, \quad y(1) = 0, \quad y'(1) = e.
\]

**Solution:** The associated auxiliary equation \( r^2 - 2r + 1 = 0 \) has a double root \( r = 1 \). Therefore, the general solution to the corresponding homogeneous equation is \( y_h = C_1 e^t + C_2 t e^t \). We seek a particular solution to the non-homogeneous equation in the form

\[
y_p = v_1(t)e^t + v_2(t)te^t.
\]

We know that \( v_1 \) and \( v_2 \) have to satisfy the following system of equations:

\[
\begin{align*}
v_1' e^t + v_2' te^t &= 0 \\
v_1' e^t + v_2'(1 + t)e^t &= t - 1 e^t.
\end{align*}
\]

The determinant of the system is the Wronskian

\[
W[e^t, te^t] = \begin{vmatrix} e^t & te^t \\ e^t & (1 + t)e^t \end{vmatrix} = e^{2t}.
\]

Using Cramer’s rule to solve for \( v_1' \) and \( v_2' \), we get

\[
v_1' = e^{-2t} \begin{vmatrix} 0 & te^t \\ t^{-1}e^t & (1 + t)e^t \end{vmatrix} = -e^{-2t} e^{2t} = -1,
\]

and

\[
v_2' = e^{-2t} \begin{vmatrix} e^t & 0 \\ e^t & t^{-1}e^t \end{vmatrix} = \frac{1}{t}.
\]

Therefore,

\[
v_1 = -t \quad \text{and} \quad v_2 = \log |t|,
\]

so \( y_p = -te^t + te^t \log |t| \). Since \( v_1 e^t = -te^t \) can be absorbed into \( y_h \), the general solution is

\[
y = C_1 e^t + C_2 te^t + te^t \log |t|.
\]

Using the initial conditions, we obtain

\[
\begin{align*}
C_1 e + C_2 e &= 0 \\
C_1 e + 2C_2 e &= e,
\end{align*}
\]

which implies \( C_1 = C_2 = 0 \). Therefore, the solution to the given initial value problem is

\[
y = te^t \log |t|.
\]
3. **(25 points)** Compute the Laplace transform of the function

\[ f(t) = te^t \cos t. \]

**Solution:**

\[
\mathcal{L}\{f\}(s) = -\frac{d}{ds}\mathcal{L}\{e^t \cos t\}(s) \quad (1)
\]

\[
= -\frac{d}{ds}\mathcal{L}\{\cos t\}(s - 1) \quad (2)
\]

\[
= -\frac{d}{ds} \frac{s - 1}{(s - 1)^2 + 1}
\]

\[
= \frac{(s - 1)^2 - 1}{[(s - 1)^2 + 1]^2}.
\]

This is valid for \( s > 1 \). In (1), we used the identity

\[
\frac{d}{ds}\mathcal{L}\{g\}(s) = -\mathcal{L}\{tg(t)\}(s),
\]

and in (2), the identity

\[
\mathcal{L}\{e^{at}g(t)\}(s) = \mathcal{L}\{g\}(s - a).
\]
4. **(25 points)** Consider the initial value problem

\[ y'' + y = 1, \quad y(0) = 0, \quad y'(0) = 0. \]  

(a) Apply the Laplace transform to both sides of the differential equation in (3) and solve for \( Y(s) \), the Laplace transform of \( y \).

(b) Solve (3) using a method of your choice.

(c) Show that the Laplace transform of the solution found in (b) equals \( Y(s) \).

**Solution:**  
(a) Since

\[ \mathcal{L}\{y'(t)\}(s) = sY(s) - y(0) = sY(s), \]

and

\[ \mathcal{L}\{y''(t)\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s), \]

applying \( \mathcal{L} \) to both sides of the ODE in (3), we get

\[ (s^2 + 1)Y(s) = \frac{1}{s}. \]

Therefore,

\[ Y(s) = \frac{1}{s(s^2 + 1)}. \]

(b) The general solution to the corresponding homogeneous equation is \( y_h = C_1 \cos t + C_2 \sin t \). Using the method of undetermined coefficients, we seek a particular solution in the form \( y_p = A \) and we find \( y_p = 1 \). Therefore, the general solution to the non-homogeneous equation is

\[ y_{\text{general}} = C_1 \cos t + C_2 \sin t + 1. \]

Now we use the initial conditions to get the system

\[
\begin{align*}
C_1 + 1 &= 0 \\
C_2 &= 0.
\end{align*}
\]

So \( C_1 = -1, \ C_2 = 0 \), and the unique solution to the IVP (3) is

\[ y = -\cos t + 1. \]

(c)

\[
\begin{align*}
\mathcal{L}\{-\cos t + 1\}(s) &= -\frac{s}{s^2 + 1} + \frac{1}{s} \\
&= \frac{1}{s(s^2 + 1)} \\
&= Y(s).
\end{align*}
\]