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Explain your work
1. **(20 points)** Consider the differential equation

\[ \frac{dy}{dt} = y^4 + 2y^3 - y^2 - 2y. \]

(a) Find all the equilibria.

(b) Sketch the phase line and classify the equilibria.

(c) Let \( y(t) \) be the solution satisfying \( 0 < y(0) < 1 \). Find the limit of \( y(t) \) as \( t \to \infty \).

**Solution:** (a) Let \( f(y) = y^4 + 2y^3 - y^2 - 2y \). Factoring \( f \), we obtain:

\[ f(y) = y(y - 1)(y + 1)(y + 2), \]

so the equilibria are \(-2, -1, 0, \) and \(1\).

(b) Computing the values of \( f \) at test points between the equilibria yields the following information:

- \( f(y) > 0 \) for \( y < -2 \);
- \( f(y) < 0 \) for \(-2 < y < -1 \);
- \( f(y) > 0 \) for \(-1 < y < 0 \);
- \( f(y) < 0 \) for \(0 < y < 1 \);
- \( f(y) > 0 \) for \( y > 1 \).

The phase line is pictured in Fig. 1. Therefore, \(-2 \) and \(0 \) are sinks and \(-1 \) and \(1 \) are sources.

(c) By (b), \( y(t) \to 0 \), as \( t \to \infty \).

![Figure 1: The phase line.](image-url)
2. (20 points) Consider the second-order differential equation
\[
\frac{d^2y}{dt^2} + 3 \frac{dy}{dt} - 10y = 0.
\]
(a) Convert into a system of first order differential equations.
(b) Compute the eigenvalues and eigenvectors of the system.
(c) Sketch the phase portrait and classify it.

**Solution:** (a) The corresponding system is
\[
\begin{align*}
\frac{dy}{dt} &= v \\
\frac{dv}{dt} &= 10y - 3v,
\end{align*}
\]
with matrix
\[
A = \begin{bmatrix} 0 & 1 \\ 10 & -3 \end{bmatrix}.
\]
(b) Since the trace is \( T = -3 \) and determinant \( D = -10 \), the eigenvalues are
\[
\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2} = -5, 2.
\]
For \( \lambda_1 = -5 \), any corresponding eigenvector \( V_1 = (a, b)^T \) satisfies
\[
\begin{bmatrix} 5 & 1 \\ * & * \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]
Thus \( 5a + b = 0 \), or \( b = -5a \) (the straight line through the origin with slope \(-5\)).
For \( \lambda_2 = 2 \), any corresponding eigenvector \( V_2 = (a, b)^T \) satisfies
\[
\begin{bmatrix} -2 & 1 \\ * & * \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]
Thus \( -2a + b = 0 \), or \( b = 2a \) (the straight line through the origin with slope \(2\)).
(c) Since \( \lambda_1 < 0 < \lambda_2 \), the phase portrait is a **saddle**. It is pictured in Fig. 2.
3. (20 points) Solve the initial-value problem

\[
\frac{dY}{dt} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} Y, \quad Y(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.
\]

Approximately sketch the solution and classify the given system.

**Solution:** The trace of the system is \( T = 0 \) and the determinant is \( D = 4 \), so the system is a center. The eigenvalues are solutions of

\[
\lambda^2 + 4 = 0,
\]

so \( \lambda = \pm 2i \). For \( \lambda = 2i \), any corresponding eigenvector \( V = (x, y)^T \) satisfies

\[
\begin{bmatrix} * & * \\ 1 & -1 - 2i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

Thus \( x - (1 + 2i)y = 0 \). Taking \( y = 1 \), we obtain

\[
V = \begin{bmatrix} 1 + 2i \\ 1 \end{bmatrix}.
\]

This gives us a complex solution

\[
Y(t) = e^{2it}V = (\cos 2t + i \sin 2t)^T \begin{bmatrix} 1 + 2i \\ 1 \end{bmatrix} = \begin{bmatrix} (\cos 2t - 2 \sin 2t) + i(2 \cos 2t + \sin 2t) \\ \cos 2t + i \sin 2t \end{bmatrix}.
\]

Separating the real and imaginary parts we obtain

\[
Y_{\text{Re}}(t) = \begin{bmatrix} \cos 2t - 2 \sin 2t \\ \cos 2t \end{bmatrix} \quad \text{and} \quad Y_{\text{Im}}(t) = \begin{bmatrix} 2 \cos 2t + \sin 2t \\ \sin 2t \end{bmatrix}.
\]
The general solution of the given system is thus

\[ Y(t) = c_1 Y_{\text{Re}}(t) + c_2 Y_{\text{Im}}(t), \]

for some real constants \( c_1, c_2 \). Taking \( t = 0 \), we obtain

\[
\begin{bmatrix} 2 \\ 0 \end{bmatrix} = c_1 Y_{\text{Re}}(0) + c_2 Y_{\text{Im}}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ c_1 \end{bmatrix}.
\]

Therefore, \( c_1 = 0 \) and \( c_2 = 1 \), so the desired solution is

\[ Y(t) = \begin{bmatrix} 2 \cos 2t + \sin 2t \\ \sin 2t \end{bmatrix}. \]

See Fig. 3.

Figure 3: The periodic solution in problem 3.
4. (20 points) Compute the solution of the following initial-value problem:

\[ \frac{d^2 y}{dt^2} + 9y = 5 \cos 2t, \quad y(0) = 2, \quad y'(0) = -6. \]

Is the harmonic oscillator resonant or non-resonant?

**Solution:** The characteristic equation is \( s^2 + 9 = 0 \), so \( s_{1,2} = \pm 3i \). Therefore, the general solution to the corresponding homogeneous equation is

\[ y_h(t) = c_1 \cos 3t + c_2 \sin 3t. \]

Since the period of the natural response is \( 2\pi/3 \) and the period of the forcing function is \( 2\pi/2 = \pi \), the system is non-resonant.

We seek a particular solution of the complexified equation

\[ \frac{d^2 y}{dt^2} + 9y = 5e^{2it} \]

in the form \( y_c(t) = ae^{2it} \), for some complex constant \( a \). Substituting into the equation, we obtain

\[ -4ae^{2it} + 9ae^{2it} = 5e^{2it}, \]

which yields \( a = 1 \). Therefore, \( y_c(t) = e^{2it} \). A particular solution to the original equation is therefore

\[ y_p(t) = \text{Re} \ y_c(t) = \cos 2t \]

and the general solution is

\[ y(t) = y_h(t) + y_p(t) = c_1 \cos 3t + c_2 \sin 3t + \cos 2t. \]

Since \( y'(t) = -3c_1 \sin 3t + 3c_2 \cos 3t - 2 \sin 2t \), using the initial conditions, we obtain

\[ \begin{align*}
  c_1 + 1 &= 2 \\
  3c_2 &= -6.
\end{align*} \]

This implies \( c_1 = 1 \) and \( c_2 = -2 \), so the desired solution is

\[ y(t) = \cos 3t - 2 \sin 3t + \cos 2t. \]
5. (20 points) Consider the following system of differential equations:

\[
\begin{align*}
\dot{x} &= x(2 - x - y) \\
\dot{y} &= y(y - x).
\end{align*}
\]

The point \((x, y) = (0, 0)\) is clearly an equilibrium.

(a) Find all the other equilibria.

(b) Linearize the system at each equilibrium found in (a) and sketch the phase portrait of the
linearized system.

(c) Classify all the equilibria of the nonlinear system found in (a).

(d) Is the system Hamiltonian?

Solution: (a) If \(y = 0\) and \(x \neq 0\), then \(2 - x - y = 0\), so \(x = 2\). Thus \((2, 0)\) is an equilibrium.

If \(y \neq 0\), then \(y = x\), so \(x \neq 0\), which implies \(2 - x - y = 0\). Since \(x = y\), it follows that \((1, 1)\)
is also an equilibrium. Since this exhausts all the possibilities, the only equilibria other than
\((0, 0)\) are \((2, 0)\) and \((1, 1)\).

(b) The vector field of the system is

\[
F(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} = \begin{bmatrix} 2x - x^2 - xy \\ -xy + y^2 \end{bmatrix}.
\]

Its derivative matrix is

\[
DF(x, y) = \begin{bmatrix} 2 - 2x - y & -x \\ -y & -x + 2y \end{bmatrix}.
\]

At \((1, 1)\) we have

\[
DF(1, 1) = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}.
\]

Since the determinant of this matrix is \(D = -2\) is negative, the linearization at \((1, 1)\) is a
saddle. See Fig. 4.

At \((2, 0)\) we have

\[
DF(2, 0) = \begin{bmatrix} -2 & -2 \\ 0 & -2 \end{bmatrix}.
\]

Since the eigenvalues of this matrix \(\lambda_1 = \lambda_2 = -2\) are negative, the linearization at \((2, 0)\) is a
(repeated eigenvalue) sink. See Fig. 4.

(c) Since the linearizations at both \((1, 1)\) and \((2, 0)\) have eigenvalues with non-zero real parts,
the qualitative behavior of the non-linear system near these equilibria is the same as that of
its linearization at the corresponding point. It follows that the non-linear system also has a
saddle at \((1, 1)\) and a sink at \((2, 0)\).

(d) The system is Hamiltonian if and only if

\[
\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial x}.
\]
Since
\[ \frac{\partial f}{\partial x} = 2 - 2x - y \neq x - 2y = -\frac{\partial g}{\partial x}, \]
it follows that the system is **not** Hamiltonian.

Figure 4: Linearizations at (1,1) and (2,0) in problem 5.

**Remark.** The phase portrait of the non-linear system is pictured in Fig. 5.

Figure 5: The phase portrait of the non-linear system.
6. (20 points) Using the Laplace transform solve the following initial-value problem:

\[
\frac{d^2 y}{dt^2} + 4y = 8, \quad y(0) = 11, \quad y'(0) = 5.
\]

**Solution:** Let \( \mathcal{L}[y] = Y \). Applying the Laplace transform to both sides of the differential equation, we obtain:

\[
s^2 Y(s) - 11s - 5 + 4Y(s) = \frac{8}{s}.
\]

It follows that

\[
Y(s) = \frac{11s + 5}{s^2 + 4} + \frac{8}{s(s^2 + 4)}.
\]

The partial fractions decomposition for the last term is

\[
\frac{8}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} = \frac{2}{s} - \frac{2s}{s^2 + 4}.
\]

Thus

\[
Y(s) = \frac{11s}{s^2 + 4} + \frac{5}{s^2 + 4} + \frac{2}{s} - \frac{2s}{s^2 + 4}
\]

\[
= 9 \frac{s}{s^2 + 4} + \frac{5}{2} \frac{s}{s^2 + 4} + \frac{2}{s}
\]

\[
= 9 \mathcal{L}[\cos 2t] + \frac{5}{2} \mathcal{L}[\sin 2t] + \mathcal{L}[2].
\]

Therefore, the solution is

\[
y(t) = 9 \cos 2t + \frac{5}{2} \sin 2t + 2.
\]