Fall 2017
MATH 42 Discrete Mathematics
Sample questions

1.
(a) Determine whether the following statement is true or false:

\[ ((-1)^2 < 1) \oplus [2^{-1} < 1]. \]

Answer: ________

(b) Let \( L(x, y) \) mean "\( x \) loves \( y \)" where the domain for both \( x \) and \( y \) consists of all people.

i. Write the English sentence "Jack loves nobody." in logical expression.

ii. Write the English sentence "Everybody loves everybody." in logical expression.

(c) Let \( P(x, z) \) mean "\( x + 2z = 3xz \)" where the domain consists of all real numbers. Determine the truth value of the following propositions.

i. \( P(2, \frac{1}{2}) \)  

ii. \( \exists z \ P(\frac{2}{3}, z) \) 

(d) Construct the truth table for the compound proposition: \((p \rightarrow r) \land (q \rightarrow r)\).

2.
(a) Justify why the following is a valid argument form by citing which rules of inference (see attached) are used in each step of your justification.

Premises:

\[ p \lor q \]

\[ \neg r \rightarrow \neg p \]

\[ \neg r \]

Conclusion: \( q \)

(b) Prove that if \( y^3 \) is odd, where \( y \) is an integer, then \( y \) is odd.

First identify the type of your proof: __________________________

Then give your proof below:

3.
(a) Let \( A = \{\{3\}, \{3, \{3\}\}, \{3\}\} \). Find the cardinality of \( A \).

(b) Let \( S = \{1, x\} \). List all the elements of the power set \( \mathcal{P}(S) \).

(c) Let \( C = \{y, a, 5\} \). Find the cardinality of the Cartesian product \( C \times C \).

(d) Consider the universal set \( U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \), and \( H = \{1, 3, 4\}, K = \{2, 3, 4, 9\}, G = \{1, 5, 8, 9\} \). List the elements in the set \( \overline{H} \cup K \cup \overline{G} \).
(e) In a survey, 28 students indicated they like American food, 15 students indicated they like Asian food, and 10 students like both. At least how many students were interviewed in the survey?

4.

(a) Find the domain and range of the function that assigns each integer its number of decimal digits.

(b) Determine whether the function \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) is onto if \( f(n) = \lfloor \frac{n}{3} \rfloor \). Justify your answer.

(c) Give an example of a function from \( \mathbb{Z} \) to \( \mathbb{Z} \) that is one-to-one but not onto.

5.

(a) List the terms \( a_{299}, a_{300}, a_{301} \) if \( a_n \) is the sum of all digits in the decimal representation of \( n \).

(b) Suppose that 5, \( a \), 33 are three consecutive terms of an arithmetic progression. Find the numerical value of \( a \).

(c) Suppose that \( a_1 = 2, a_2 = 8, a_3 = 26, a_4 = 80, a_5 = 242 \). Find a closed formula for the general term \( a_n \).

6.

(a) Use modulo arithmetic to find the remainder when \( 7^{98} \) is divided by 16.

(b) Given two positive integers \( 2^4 \cdot 3^2 \cdot 7^2 \cdot 13^4 \) and \( 2^5 \cdot 5 \cdot 7^2 \cdot 11 \), find the lcm.

(c) Is 770! divisible by \( 7^{128} \)? Justify your answer for full credit.

(d) Use Euclidean Algorithm to find the gcd of 321 and 2241. Show your steps for full credit.

7.

Use the following format of mathematical induction proof to show that for every positive integer \( n \),

\[
1 \cdot 2 + 2 \cdot 3 + \cdots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2).
\]

(a) Statement step:

(b) Basic step:

(c) Induction step:

(d) Conclusion step:

8.

(a) How many bit strings of length 8 contain exactly 4 1s?
(b) The English alphabet contains 21 consonants and 5 vowels. How many strings of 7 different letters of the English alphabet contain exactly 5 consonants?

(c) How many ways are there for 5 men and 8 women to stand in a line so that NO two men stand next to each other?

(d) How many positive integers between 200 and 888 inclusive are divisible by 6 or 9?

(e) How many bit strings of length 8 contain 4 consecutive 0s?

9.
Pick 36 different integers between 11 and 80 inclusive. Use Pigeonhole Principle to explain why there is always a pair of integers with a sum of 91.

(a) What are the pigeons? How many?

(b) What are the pigeonholes? How many?

(c) Then ...

10.

(a) What is the probability that when a coin is flipped five times in a row, it lands tails up every time?

(b) What is the probability that a five-card poker hand contains cards of four different suits?

(c) What is the probability that a die never comes up a three when it is rolled 4 times?

(d) What is the probability that in a group of 5 people chosen at random, there are at least two born on the same day of the week?

(e) Which is more likely: (i) a three comes up at least once when a die is rolled four times (ii) a double five comes up at least once when a pair of dice is rolled 24 times? Why?

11.

(a) Let \( A = \{1, 2, 3, 4, 5\} \) and a relation on \( A \) be defined as \( R = \{(a, b)|a \text{ divides } b\} \). List all the ordered pairs of \( R \).

(b) Define a relation \( S \) on the set of real numbers as follows: \((x, y) \in S \) if and only if \( x \geq y^2 \). Is \( S \) antisymmetric? Why?

(c) Consider the relation \( T = \{(1, 1), (2, 3), (3, 3), (1, 2)(1, 3), (2, 1)(3, 1)\} \) on the set \( \{1, 2, 3\} \). Explain why \( T \) is not an equivalence relation. Show all your work for full credit.
(d) Given an equivalence relation \( U \) on the set \( \mathbb{R} \) of real numbers as follows:

\[
(x, y) \in U \quad \text{if and only if} \quad x^2 + x = y^2 + y
\]

Find the elements of the equivalence class \([0]_U\).

12.

(a) Solve the Boolean equation: \( x + yz + \overline{x}yz = 0 \).

(b) Find the sum-of-product expansion of the Boolean function \( F(x, y, z) \) that has the value 1 if and only if \( x(y + z) = y(x + z) \). [NO need to simplify your answer.]