Problem Set 1
Math 238

1. Show that
\[ \cos \theta + \cos(\theta + x) + \cdots + \cos(\theta + nx) = \frac{\sin(n+1)x}{\sin \frac{x}{2}} \cdot \cos(\theta + \frac{1}{2}nx) \]

2. Prove that for \( m = 2, 3, \ldots \)
\[ \sin \left( \frac{\pi}{m} \right) \sin \left( \frac{2\pi}{m} \right) \sin \left( \frac{3\pi}{m} \right) \cdots \sin \left[ \frac{(m-1)\pi}{m} \right] = \frac{m}{2^{m-1}} \]

3. Describe geometrically the region in the complex plane determined by
\[ |z + i| < |z - i|, \text{ where } z = x + iy. \]

4. Prove the Minkowski inequality
\[ \sqrt{\sum_{j=1}^{n} |a_j + b_j|^2} \leq \sqrt{\sum_{j=1}^{n} |a_j|^2} + \sqrt{\sum_{j=1}^{n} |b_j|^2} \]
where \( a_j, b_j \in \mathbb{C}. \)
[Hint: Use Cauchy's inequality.]

5. Suppose \( \text{Im} z > 0. \) Show that
\[ \left| \frac{z - \alpha}{z - \overline{\alpha}} \right| = k > 0, \ \text{where } z = x + iy, \ \alpha = a + ib, \]
is an equation of a circle. Find its center and radius.
1. Let function $f$ be defined by $f(z) = u(x,y) + i v(x,y)$ and suppose that $f$ is analytic in a domain $D$. Show that

$$\left(\frac{\partial}{\partial x}|f(z)|^2 + \frac{\partial}{\partial y}|f(z)|^2\right)^2 = |f'(z)|^2 \quad (\forall z \in D).$$

2. Let $u = u(x,y)$ and $v = v(x,y)$ be harmonic functions in a domain $D$. Prove that the function

$$F = (u_y - v_x) + i (u_x + v_y)$$

is analytic in $D$. Assume $f(z) = u + iv$ is analytic in $D$.

3. Suppose $f(z)$ is analytic in a domain $D$. Suppose further that for all $z \in D$, $|f(z)| < 1$ and $f'(z) \neq 0$. Show that the function $w$ defined by

$$w(z) = \ln \left( \frac{|f'(z)|}{1 - |f(z)|^2} \right)$$

satisfies in $D$ the P.D.E.

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 4 \cdot e^{2w}.$$ 

4. Suppose $f(z)$ is analytic in $D$ and $f'(z) \neq 0$ for all $z$ in $D$. Show that

$$\nabla^2 |f(z)| = \frac{|f'(z)|^2}{|f(z)|^2}.$$
Problem Set 3

1. Show that if \( f(z) \) and \( \overline{f(z)} \) are both analytic in a domain \( D \) then \( f(z) \) is a constant.

2. If \( f(z) \) is analytic and \( |f(z)| \) is constant in \( D \) then \( f(z) \) is also a constant in \( D \).

3. Prove that \( f(z) \) is analytic if and only if \( \overline{f(z)} \) is analytic.

4. Let \( P_r(\theta) = \frac{1-r^2}{1+r^2-2r \cos \theta} \).

   Show that \( P_r(\theta) \) is harmonic in \(|z|<1\).

5. Evaluate the integral

   \[
   \int_0^{2\pi} \frac{R^2-r^2}{R^2+r^2-2r R \cos(\theta-\phi)} \, d\phi, \quad 0 \leq \theta \leq 2\pi.
   \]

   Do this problem in two ways.

6. If \( u-v = (x-y)(x^2+4xy+y^2) \) and \( f(z)=u+iv \) is an analytic function of \( z=x+iy \), find \( f(z) \) in terms of \( z \).

7. Let \( f(z) \) be analytic in a domain \( D \).

   If \( h(xy)=u(xy)v(xy) \), where \( f(z)=u+iv \), is it true that \( \nabla^2 h(xy)=0 \)? Justify your answer.
Problem Set 4

1. Let \( f(z) \) be an entire function such that 
\[ f(z) = f(z+2) \quad \text{and} \quad f(z) = f(z+i) \]
for all \( z \). Show that \( f(z) \) is a constant.

2. Let \( f(z) \) be an entire function such that 
\[ |f(z)| \leq e^x, \quad \text{where} \quad z = x + iy. \]
What can be said about \( f(z) \)? More generally, let \( f(z), g(z) \) 
be two entire functions satisfying 
\[ |f(z)| \leq |g(z)| \]
everywhere. What conclusion can you draw about 
\( f(z) \)? (Assume \( g(z) \neq 0 \)).

3. Prove or disprove that if \( f(z) \) is an entire function which does not 
vanish in the extended complex plane then \( f(z) \) is reduced to a 
constant.

4. Let \( f(z) \) be analytic on \( |z| \leq 1 \) and \( |f(z)| < 1 \) 
for \( |z| = 1 \). How many solutions does the equation 
\( f(z) = z \) have in \( |z| < 1 \)?

5. Let \( f(z) \) be analytic in a domain \( D \) and on its boundary \( C \). If \( |f(z)| \) is constant on \( C \), show that there exists at least one zero of \( f(z) \) in \( D \) unless \( f(z) \) reduces to a constant in \( D \).
6. Prove that if \( P(z) = a_0 + a_1 z + \cdots + a_n z^n \)
with \( 0 < a_0 < a_1 < \cdots < a_n \), then \( P(z) \)
has all of its zeros inside the unit circle.

7. Let \( f(z) \) be an analytic function inside and on a simple closed contour \( C \) with zeros \( a_i \), \( i = 1, 2, 3, \ldots, N \). If \( g(z) \) is analytic inside and on \( C \), then show that
\[
\frac{1}{2\pi i} \int_C g(z) \frac{f'(z)}{f(z)} \, dz = \sum_{i=1}^{N} g(a_i).
\]

8. Let \( f(z) \) be analytic in \( |z| < R \) for \( |z| = 1 \) and \( f(0) = 1 \). Prove that \( f(z) \) has at least one zero in \( |z| < 1 \).

9. If \( P(z) \) is a polynomial and \( C : |z-a| = R \), show that
\[
\int_C P(z) \, dz = -2\pi i R^2 P'(a).
\]

10. Let \( f(z) \) be analytic in \( |z| < r \). Let \( L \) denote the length of the image of \( |z| = r \) under the mapping \( w = f(z) \). Show that \( L \geq 2\pi r |f'(0)| \).
Problem Set 5

1. If \( f(z) \) is analytic in \( |z| \leq 1 \) and \( |f(z)| < 1 \) on \( |z|=1 \). Prove that there exists a point \( z_0 \) in \( |z| < 1 \) such that \( f(z_0) = z_0 \). [Hint: Use Rouche's theorem.]

2. Find all functions \( f(z) \) which are analytic in the region \( |z| \leq 1 \) and are such that (i) \( f(0) = 3 \) and (ii) \( |f(z)| \leq 3 \) for all \( z \) such that \( |z| \leq 1 \).

3. Prove that all the roots of \( z^5 + z - 16 = 0 \) lie between the circles \( |z| = 1 \) and \( |z| = 2 \).

4. Suppose that a function \( f(z) \) is analytic in \( |z| \leq 1 \) and satisfying the conditions: \( |f(e^{i\theta})| = 2 \) for all \( \theta, 0 \leq \theta \leq 2\pi \), and \( f(\frac{1}{2}) = 1 + i \). What can be said about \( f(z) \)?

5. Prove that all the roots of \( z^7 - 5z^3 + 12 = 0 \) lie between \( |z| = 1 \) and \( |z| = 2 \).

6. Find the maximum and minimum moduli of \( z^2 - z \) in the disc: \( |z| \leq 1 \).