Let \((\dagger\eta)\) denote the following proposition:

There exists a regular uncountable cardinal \(\kappa\) and set of ordinals \(C_\delta\), for \(\delta < \eta\), such that

\[
\begin{align*}
\bullet \ \text{sup}(C_\delta) & \text{ is a regular cardinal greater than } \kappa; \\
\bullet \ C_\delta & \text{ is closed in } \text{sup}(C_\delta); \text{ and} \\
\bullet \ \text{if } \delta \neq \delta' \text{ and } \alpha \in C_\delta \text{ and } \alpha' \in C_{\delta'} \text{ and } \text{cf}(\alpha) = \text{cf}(\alpha') = \kappa, \text{ then } \text{cf}^L(\alpha) \neq \text{cf}^L(\alpha').
\end{align*}
\]

The purpose of this note is proving the following three claims.

1. If \(N < \omega\), then \((\dagger N)\) is consistent, relative to the consistency of ZFC.
2. If \((\dagger \omega)\), then \(0^\#\) exists.
3. If \(0^\#\) exists, then \((\dagger \eta)\) holds for all \(\eta\).

The last of these is easy. Let \(I\) be the club class of Silver indiscernibles provided by \(0^\#\), let \(\kappa = \omega_1\), and let \(C_\delta = I \cap [\omega_\delta + 1, \omega_{\delta+2})\), for \(\delta < \eta\).

Claim (2) uses Strong Covering and a bit of Jensen’s fine structure theory for \(L\). Claim (1) is proved using forcing.

Some non-standard notation will be useful. If \(X\) is a set of ordinals and \(\lambda\) is regular, let

\[
\begin{align*}
\text{lim}(X) &= \left\{ \alpha \in X : \alpha = \text{sup}(X \cap \alpha) \right\} \\
X^{(\lambda)} &= \left\{ \alpha \in \text{lim}(X) : \text{cf}(\alpha) = \lambda \right\}.
\end{align*}
\]

If \(\lambda\) is a cardinal, then \(H_\lambda\) is a the set of all sets hereditarily of cardinality less than \(\lambda\). Without further mention, let us fix a well ordering \(\triangleleft_\lambda\) of \(H_\lambda\) and write \(\langle H_\lambda \rangle\) to indicate the structure \((H_\lambda; \triangleleft_\lambda, \in)\). This equips \(H_\lambda\) with definable Skolem functions, so that we may unambiguously take Skolem hulls of subsets of \(H_\lambda\). It also allows us to make canonical choices when constructing sequences of elements of \(H_\lambda\).

This is version 1.0
1. $(\mathfrak{t}_\omega)$ implies that $0^\#$ exists

**Theorem 1.** If $0^\#$ does not exist, then $(\mathfrak{t}_\omega)$ fails.

**Proof:** For a contradiction, assume that $(\mathfrak{t}_\omega)$ holds, as witnessed by the regular uncountable cardinal $\kappa$ and the closed sets $C_n$, for $n < \omega$. Set $\kappa_n = \sup(C_n)$. Then $\kappa_n > \kappa$ is a regular cardinal. Let $\lambda > \sup_{n<\omega} \kappa_n$ be regular. By Strong Covering, there exists an unbounded $C \subseteq [\lambda]^\kappa$ such that $C \in L$ and $C$ is closed under unions of chains of uncountable cofinality.

Define $X_i$ and $Y_i$, for $i \leq \kappa$, as follows:

$$X_0 = \kappa \cup \{ C_n : n \in \omega \};$$

$$Y_0 = \kappa;$$

$$X_{i+1} = \text{least } X < H_\lambda \text{ such that } X_i \cup Y_i \cup \left\{ \sup(Y_i \cap \kappa_n) : n \in \omega \right\} \subseteq X;$$

$$Y_{i+1} = \text{least } Y \in C \text{ such that } Y \supseteq X_{i+1} \cap \lambda;$$

$$X_i = \bigcup_{j<i} X_j, \text{ if } i \text{ is a limit ordinal;} \text{ and}$$

$$Y_i = \bigcup_{j<i} Y_j, \text{ if } i \text{ is a limit ordinal.}$$

Then $X_\kappa < H_\lambda$ has cardinality $\kappa$. Also $\text{cf}(\sup(X_\kappa \cap \kappa_n)) = \kappa$ because $\sup(X_\kappa \cap \kappa_n) = \sup_{i<\kappa}(\sup(X_i \cap \kappa_n))$ and $\sup(X_i \cap \kappa_n) \leq \sup(Y_i \cap \kappa_n) < \sup(X_{i+1} \cap \kappa_n)$, for $i \leq \kappa$. And $\sup(X_\kappa \cap \kappa_n) \in C_n$, since $C_n \in X_\kappa$. Furthermore $Y_\kappa \in C$, since $\kappa$ has uncountable cofinality. It follows that $\text{cf}^L(\sup(Y_\kappa \cap \kappa_n)) \neq \text{cf}^L(\sup(Y_{\kappa'} \cap \kappa_m))$, if $m \neq n$.

If $M$ is the smallest elementary substructure of $L_\lambda$ such that $Y_\kappa \subseteq M$, then $M \cap \lambda = Y_\kappa$, since $L_\lambda$ is definable over $H_\lambda$. Let $J_\alpha$ be the transitive collapse of $M$. To reach a contradiction, it suffices to establish the following

**Lemma.** $\left\{ \text{cf}^L(\tau) : |\alpha|^L < \tau < \alpha \land \tau \text{ is a regular cardinal in } J_\alpha \right\}$ is finite.

**Proof:** Work in $L$ and fix $\alpha$. For $J_\alpha$-regular $\tau$ such that $|\alpha| < \tau < \alpha$, let $\beta(\tau)$ be the least $\beta$ such that $\omega^{\beta(\tau)}_{n+1} < \tau$, for some $n > 0$, and let $n(\tau)$ be the least $n$ such that $\omega^{\beta(\tau)}_{n+1} < \tau$. Now if $|\alpha| < \tau < \nu < \alpha$ and both $\tau$ and $\nu$ are $J_\alpha$-regular, then $\beta(\tau) \geq \beta(\nu)$. And if $\beta(\tau) = \beta(\nu)$, then $n(\tau) = n(\nu)$. Thus, it suffices to see that if $\beta(\nu) = \beta(\nu)$ and $n(\tau) = n(\nu)$, then $\text{cf}(\tau) = \text{cf}(\nu)$.

The $\Sigma^\beta_\alpha$-cofinality of an ordinal $\delta \leq \beta$ is defined to be the least ordinal $\gamma$ such that there exists a $\Sigma^\beta_\alpha(J_\beta)$ definable function with domain equal to $\gamma$ and range cofinal in $\delta$. Let $\Sigma^\beta_\alpha$-cf denote this ordinal $\gamma$.

**Claim.** Suppose that $\tau$ is a regular cardinal in $J_\beta$ and that $\omega^{\beta}_{n+1} < \tau \leq \omega^{\beta}_{n}$. Set $\sigma = \min(\omega^{\beta}_{n+1}, \Sigma^\beta_{n+1}$-cf$(\omega^{\beta}_{n}))$. Then there exists a monotonically non-decreasing function $f: \sigma \rightarrow \tau$ such that $\text{rng}(f)$ is cofinal in $\tau$. Hence $\text{cf}(\sigma) = \text{cf}(\tau)$.

**Proof:** Set $\hat{\rho} = \rho^n$ and $\rho = \rho^{\beta}_{n+1}$. Let $A$ be a $\Sigma_n$-master code for $J_\beta$. Set $\mathfrak{a} = \langle J_\beta, A \rangle$. Then $\sigma = \min(\omega\hat{\rho}, \Sigma^\beta_1$-cf$(\omega\hat{\rho}))$.  

2
1. ($\uparrow_\omega$) IMPLIES THAT $0^\#$ EXISTS

Fix a $J_\beta$-regular cardinal $\tau$ such that $\omega \rho < \tau \leq \omega \dot{\rho}$. Then $\tau$ is regular in $\mathfrak{A}$.

Because $\rho = \rho_1^\mathfrak{A}$, there exists a $\Sigma_1(\mathfrak{A})$ definable function $g$ such that $\text{dom}(g) \subseteq \omega \rho$ and $\text{rng}(g) = \omega \dot{\rho}$. Let $\varphi$ be a $\Sigma_0(\mathfrak{A})$ formula such that $g(\xi) = \eta$ iff $\mathfrak{A} \models \exists z \varphi(\xi, \eta, z)$.

For $\delta < \omega \dot{\rho}$, set

$$g^\delta = \{ (\xi, \eta) \in \omega \rho \times \delta : \exists z \in S_\delta \varphi(\xi, \eta, z) \}.$$  

Then $g^\delta \in J_\dot{\rho}$ and $g = \bigcup_{\delta < \omega \dot{\rho}} g^\delta$.

Note next that there exists a $\Sigma_1(\mathfrak{A})$-definable function $h$ such that

(a) $\text{dom}(h)$ is an unbounded subset of $\sigma$,
(b) $\text{rng}(h)$ is an unbounded subset of $\omega \dot{\rho}$, and
(c) if $\gamma < \sigma$, then $\sup(\text{rng}(h \upharpoonright \gamma)) < \omega \dot{\rho}$.

If $\sigma = \Sigma_1^0\text{-cf}(\omega \dot{\rho})$, then there exists such a function with $\text{dom}(h) = \sigma$. Otherwise, set $h = g$. To see (c) in this case, let $\gamma < \sigma = \omega \rho$. Then $\text{dom}(h \upharpoonright \gamma) \in J_\dot{\rho}$. Thus

$$\tilde{h}(\delta) = \begin{cases} h(\delta), & \text{if } \delta \in \text{dom}(h) \cap \gamma \\ 0, & \text{if } \delta \in \gamma \setminus \text{dom}(h) \end{cases}$$

is a $\Sigma_1(\mathfrak{A})$-definable function with domain $\gamma$. But then $\text{rng}(h \upharpoonright \gamma) \subseteq \text{rng}(\tilde{h})$ is bounded below $\omega \dot{\rho}$, since $\gamma < \sigma \leq \Sigma_1^0\text{-cf}(\omega \dot{\rho})$.

Define $f: \sigma \rightarrow \tau$ by

$$f(\gamma) = \sup_{\xi \in \text{dom}(h \upharpoonright \gamma)} \sup \left( \text{rng}(g^{h(\xi)} \cap \tau) \right).$$

Clearly $f$ is monotonically non-decreasing.

If $\gamma < \sigma$, then $\delta = \sup(\text{rng}(h \upharpoonright \gamma)) < \omega \dot{\rho}$, hence $f(\gamma) \leq \sup(\text{rng}(g^\delta \cap \tau))$. And $\text{rng}(g^\delta)$ is bounded below $\tau$, since $g^\delta \in J_\dot{\rho}$ and $\tau$ is regular in $\mathfrak{A}$. Hence $f(\gamma) < \tau$, for all $\gamma < \sigma$.

Finally, note that

$$\sup \left( \text{rng}(f) \right) = \sup_{\xi \in \text{dom}(h)} \sup \left( \text{rng}(g^{h(\xi)} \cap \tau) \right) = \sup \left( \text{rng}(g) \cap \tau \right) = \tau. \quad \square$$

2. The consistency of ($\uparrow_N$)

The goal of this section is proving the consistency of ($\uparrow_N$) for finite $N$. In fact, we shall prove

**Theorem 2.** Fix $N$ such that $1 \leq N < \omega$. Relative to the consistency of ZFC, it is consistent that there exist club sets $K_n \subseteq \omega_{n+2}$, for $n < N$, such that if $\alpha \in K_n^{(\omega_1)}$, then $\text{cf}^L(\alpha) = \omega_n^{L+2}$.

It is for convenience (uniform indexing at different stages in the construction) that none of the $K_n$'s have cofinality $\omega_1$ limit points of $L$-cofinality $\omega_1$. Essentially the same proof shows that it is consistent that there exist club sets $K_n \subseteq \omega_{n+2}$, for $n < N$, such that if $\alpha \in K_n^{(\omega_1)}$, then $\text{cf}^L(\alpha) = \omega_n^{L+1}$.
2. THE CONSISTENCY OF \((\dagger_N)\)

The club sets \(K_n\) will be added using an \(N + 1\) stage iteration over \(L\), indexed by \(I\) for \(1 \leq I \leq N + 1\). At stage 1, \(\omega_{N+1}^L\) is collapsed to cardinality \(\omega_1\). At stage \(I > 1\), the club set \(K_{I-2}\) is added. Doing this requires collapsing \(L\)-cardinals between what are ultimately the \(\omega_I\)'s, as well as adding auxiliary club subsets to these collapsed \(L\)-cardinals.

In the ultimate extension we shall have that \(\omega_i = \omega_{n(i)}^L\), for \(i \leq N + 1\), where

\[
\begin{align*}
n(0) &= 0 \\
n(1) &= 1 \\
n(i) &= n(i - 1) + N + 3 - i, \quad \text{for} \ 2 \leq i \leq N + 1
\end{align*}
\]

In particular, note that \(\omega_i^L\) will be preserved.

The model constructed is \(L[G_1 \cdots G_{N+1}]\), where \(G_1\) is generic over \(L\) and \(G_I\) is generic over \(L[G_1 \cdots G_{I-1}]\), if \(I > 1\). At the beginning of stage \(I\), where \(1 \leq I \leq N + 1\), in the model \(L[G_1 \cdots G_{I-1}]\) \((L, \text{if } I = 1)\), the following five recursion hypotheses hold.

(A) \(\omega_i = \omega_{n(i)}^L\), for \(i \leq I\).

(B) All \(L\)-cardinals greater than or equal to \(\omega_I\) are cardinals.

It follows from (A) and (B) that

\[
\omega_j = \omega_{n(i) - (i - 1) + (i - 1)}^L
\]

(C) The GCH holds.

If \(I > 1\), let \(\lambda^0, \ldots, \lambda^{N+I-1}\) enumerate the \(L\)-cardinals strictly between \(\omega_{I-1}\) and \(\omega_I\) in decreasing order. Then \(\lambda^m = \omega_{n(I-1)+N-I+2-m}^L\).

(D) If \(I > 1\), then there exists a club \(C_m \subseteq \lambda^m\), for \(0 \leq m \leq N - I + 1\), such that \(\text{ot}(C_m) = \omega_{I-1}\), and such that, if \(\alpha \in C_m^{(\omega_1)}\), then \(\text{cf}^L(\alpha) = \omega_{N+1-m}^L\).

The last clause in (D) is trivial if \(I = 2\) because \(\text{ot}(C_m) = \omega_1\) in that case.

(E) If \(\kappa \geq \omega_I\) is regular, then \(\{ \alpha < \kappa : \text{cf}^L(\alpha) = \lambda^m \}\) is stationary in \(\kappa\), for \(0 \leq m \leq N - I + 1\).

The partial ordering \(\mathbb{C}(\kappa, \lambda, \eta)\) is the main building block for the proof. If \(\lambda < \kappa\) are regular cardinals and \(\eta < \lambda\) is \(L\)-regular, then \(\mathbb{C}(\kappa, \lambda, \eta)\) is a forcing property designed to add an order type \(\lambda\) club subset \(C\) of \(\kappa\) such that \(C^{(\omega_1)} \subseteq \{ \alpha < \kappa : \text{cf}^L(\alpha) = \eta \}\).

Declare that \(c \in \mathbb{C}(\kappa, \lambda, \eta)\) iff

- \(c\) is a closed subset of \(\kappa\). (So \(\sup(c) \in c\), if \(c \neq \emptyset\).)
- \(\text{ot}(c) < \lambda\)
- If \(\alpha \in c^{(\omega_1)}\), then \(\text{cf}^L(\alpha) = \eta\).

Conditions in \(\mathbb{C}(\kappa, \lambda, \eta)\) are inversely ordered by end-extension.

Note that forcing with \(\mathbb{C}(\kappa, \lambda, \eta)\) adds a club subset of \(\kappa\) having order type at most \(\lambda\) and containing only cofinality \(\omega_1\) limit points of \(L\)-cofinality \(\eta\). Thus in an extension in which there exists a \(\mathbb{C}(\kappa, \lambda, \eta)\) generic object and in which there are no new sets of fewer than \(\lambda\) many ordinals, there exists an order type \(\lambda\) club subset \(C\) of \(\kappa\) such that \(C^{(\omega_1)} \subseteq \{ \alpha < \kappa : \text{cf}^L(\alpha) = \eta \}\).
2. The Consistency of \((\uparrow_1)\)

**Stage 1.** Let \(G_1\) be generic over \(L\) for the simple collapse of \(\omega_{N+1}\) to cardinality \(\omega_1\), using countable conditions. In \(L[G_1]\) we have that \(\omega_1 = \omega^L_{\kappa_1} = \omega^L_{\kappa_1(1)}\) and \(\omega_2 = \omega^L_{N+2} = \omega^L_{n(2)}\) and that \(L\)-cardinals greater than or equal to \(\omega^L_{N+2}\) are preserved. Hence (A) and (B) hold. (C) is standard; (D) is trivial; and (E) holds because this forcing has cardinality \(\omega_{N+1}\) in \(L\). Hence for regular \(\kappa \geq \omega_2 = \omega^L_{N+2}\), every club set of \(\kappa\) that lies in the extension contains a ground model club set.

**Stage I \((1 < I < N + 1)\).** Work in \(L[G_1 \cdots \cdots G_{I-1}]\). Let

\[
C = \prod_{m=0}^{N-I+1} C(\omega^I_{m+1}, \omega^I_m, \omega^L_{I+m})
\]

and let \(G_I\) be \(C\) generic over \(L[G_1 \cdots \cdots G_{I-1}]\).

The main fact to be proved is this

**Claim.** \(C\) is \(\omega_I\)-distributive.

Before proving the Claim, let us use it to show that hypotheses (A)–(E) are maintained at the beginning of stage \(I + 1\) and that \(K_{I-2}\) as in the statement of the Theorem is added at stage \(I\).

Since forcing with \(C\) introduces a club subset of each of \(\omega^I_{I+1}, \ldots, \omega^I_{N+1}\) having order type at most \(\omega_I\), it is clear that forcing with \(C\) collapses all of \(\omega^I_{I+1}, \ldots, \omega^I_{N+1}\). By the Claim, no cardinals less than or equal to \(\omega_I\) are collapsed. And cardinals greater than or equal to \(\omega^I_{N+2}\) are preserved, since \(|C| = \omega^I_{N+1}\). Thus to establish (A) and (B), it suffices to unwind the numerology and see that \(\omega^I_{N+2} = \omega^L_{n(I+1)}\).

\[
\begin{align*}
\omega^I_{N+2} &= \omega^L_{n((N+2)-(N+2)-I)-(N+2)-I)} \\
&= \omega^L_{n(I)+N+2-I} \\
&= \omega^L_{I+n(I)+N+3-(I+1)} \\
&= \omega^L_{n(I+1)}
\end{align*}
\]

by (A) and (B) at stage \(I\)

since \(I < N + 2\)

Hypothesis (C) is easy using the Claim and that \(|C| = \omega^I_{N+1} < \omega^L_{I+1} \downarrow \cdots \downarrow G_1\) 

Also since \(|C| = \omega^I_{N+1} < \omega^L_{I+1} \downarrow \cdots \downarrow G_1\) in \(L[G_1 \cdots \cdots G_I]\) every club subset of a regular \(\kappa \geq \omega^I_{I+1}\) contains a ground model club set. This observation suffices to see that Hypothesis (E) is maintained at the beginning of stage \(I + 1\).

Finally, to verify that (D) holds at the beginning of stage \(I + 1\), suppose that \(0 \leq m \leq N - (I + 1) + 1 = N - I\). Let \(\lambda^0, \ldots, \lambda^{N-I}\) be as defined at the beginning of stage \(I + 1\). Then \(\lambda^m = \omega^L_{n(I+1)-m}\) (in the current model, namely \(L[G_1 \cdots \cdots G_{I-1}]\)). Set \(j = N - I + 1 - m\). Then \(1 \leq j \leq N - I + 1\), so forcing with \(C\) adds a \(C(\omega_{I+j}, \omega_I, \omega^I_{I+j})\) generic object. And \(I + j = N + 1 - m\). Hence in \(L[G_1 \cdots \cdots G_I]\) there exists a club \(C_m \subseteq \lambda^m\) such that \(\text{ot}(C_m) = \omega_I\) (using the Claim) and such that every \(\alpha \in C_m^{(\omega_I)}\) has \(L\)-cofinality \(\omega^I_{N+1-m}\).

Forcing with \(C\) also adds a \(C(\omega_I, \omega_I, \omega^I_I)\) generic object. Hence in \(L[G_1 \cdots \cdots G_I]\) there exists a club \(K_{I-2} \subseteq \omega_I\) such that if \(\alpha \in K_{I-2}^{(\omega_I)}\), then \(\text{ct}^L(\alpha) = \omega^I_I\).
Thus proving the Claim suffices to prove the Theorem.

**Proof that \( \mathbb{C} \) is \(<\omega_I\)-distributive:** Fix a condition \( \bar{c} \in \mathbb{C} \) and a sequence \( \vec{D} = \langle D_i : i < \omega_{I-1} \rangle \) of dense subsets of \( \mathbb{C} \). Fix \( \lambda \) large and regular and define a sequence \( \langle M^m : 0 \leq m \leq N - I + 1 \rangle \) of elementary substructures of \( H_\lambda \) by recursion on \( m \). Our goal is to construct \( M = M^{N-I+1} \), an elementary substructure of \( H_\lambda \) of cardinality \( \omega_{I-1} \) such that \( cf^L(\text{sup}(M \cap \omega_{N+1-m})) = \lambda^m \), for \( 0 \leq m \leq N - I + 1 \). Using Hypothesis (D) we can then build a suitable continuous tower \( \langle N_i : i \leq \omega_{I-1} \rangle \) inside \( M \) and follow it to build a condition \( c \) extending \( \bar{c} \) and meeting each of the \( D_i \)'s.

Let \( C_0, \ldots, C_{N-I+1} \) be as in Hypothesis (D).

We shall define \( M^m \prec H_\lambda \) by recursion on \( m \). Simultaneously, setting \( \beta^m = \sup(M^m \cap \omega_{N+1-m}) \), we shall define a club subset \( E_m \) of \( \beta^m \) such that \( E_m \subseteq M^m \), for all \( m \). Begin by choosing \( M^0 \prec H_\lambda \) such that

- \( \lambda^0 \cup \omega_N \subseteq M^0 \) and \( |M^0| = \omega_N \);
- \( \bar{c}, \mathbb{C}, \vec{D}, C_0, \ldots, C_{N-I+1} \subseteq M^0 \);
- \( [M^0]^\omega_I \subseteq M^0 \); and
- \( cf^L(\beta^0) = \lambda^0 \) (hence \( cf(\beta^0) = \omega_{I-1} \)), where \( \beta^0 = M^0 \cap \omega_{N+1} \).

To see that this is possible, define a continuous tower \( \langle M_\xi : \xi < \omega_{N+1} \rangle \) as follows:

\[
M_0 = \text{least } M \prec H_\lambda \text{ such that } \\
\lambda^0 \cup \omega_N \cup \{ \bar{c}, \mathbb{C}, \vec{D}, C_0, \ldots, C_{N-I+1} \} \subseteq M \\
M_{\xi+1} = \text{least } M \prec H_\lambda \text{ such that } [M_\xi]^\omega_I \cup \{ M_\xi \} \subseteq M \\
M_\xi = \bigcup_{\zeta < \xi} M_\zeta \text{, if } \xi \text{ is a limit ordinal.}
\]

(By induction, \(|M_\xi| = \omega_N \) and \( I - 2 < N \), so \(|[M_\xi]^\omega_I = \omega_N \).)

Now \( \{ M_\xi \cap \omega_{N+1} : \xi < \omega_{N+1} \} \) is club in \( \omega_{N+1} \). Using Hypothesis (E), there exists a limit ordinal \( \xi \) such that \( \text{sup}(M_\xi \cap \omega_{N+1}) \) has \( L \)-cofinality \( \lambda^0 \), hence cofinality \( \omega_{I-1} \). Then \( [M_\xi]^\omega_I \subseteq M_\xi \), because \( \omega_{I-1} \) is regular. Set \( M^0 = M_\xi \).

Next, let \( f \in M^0 \cap L \) be an increasing, continuous, cofinal function from \( \lambda^0 \) into \( \beta^0 \).

Set \( E_0 = f'' C_0 \), then \( f \in E_0 \cap L \), and, using that \( f \in L \), if \( \alpha \in E_0(\omega_I) \), then \( cf^L(\alpha) = \omega_{N+1} \).

Choose \( M^{m+1} \prec M^m \) such that

- \( \lambda^0 \cup \omega_{-(m+1)} \subseteq M^{m+1} \) and \( |M^{m+1}| = \omega_{-(m+1)} \);
- \( \bar{c}, \mathbb{C}, \vec{D}, C_0, \ldots, C_{N-I+1} \subseteq M^{m+1} \);
- \( E_0 \cup \cdots \cup E_m \subseteq M^{m+1} \);
- \( [M^{m+1}]^\omega_I \subseteq M^{m+1} \); and
- \( cf^L(\beta^{m+1}) = \lambda^{m+1} \) (hence \( cf(\beta^{m+1}) = \omega_{I-1} \)), where \( \beta^{m+1} = M^{m+1} \cap \omega_{N-m} \).

Again this is possible using Hypothesis (E).

Let \( f \in M^{m+1} \cap L \) be an increasing, continuous, cofinal function from \( \lambda^{m+1} \) into \( \beta^{m+1} \) and set \( E_{m+1} = f'' C_{m+1} \). Then \( E_{m+1} \) is a club subset of \( \beta^{m+1} \) having order type \( \omega_{I-1} \) and the property that if \( \alpha \in E_{m+1}(\omega_I) \), then \( cf^L(\alpha) = \omega_{N+1-(m+1)} \).

Finally, set \( M = M^{N-I+1} \). Then we have

- \( M \prec H_\lambda \) and \( |M| = \omega_{I-1} \);
2. THE CONSISTENCY OF \( \langle \uparrow N \rangle \)

- \( E_m \subseteq M \), for all \( m \), where \( E_m \) is club in \( \text{sup}(E_m) = \beta^m \), has order type \( \omega_{I-1} \), and has the property that if \( \alpha \in E_m^{\omega} \), then \( \text{cf}^L(\alpha) = \omega_{N+1-m} \); and
- \( \beta^m = \text{sup}(M \cap \omega_{N+1-m}) \) and \( \text{cf}^L(\beta^m) = \lambda^m \), for all \( m \).

Next, let us define a continuous tower \( \langle N_i : i \leq \omega_{I-1} \rangle \) of elementary substructures of \( M \). We shall build a condition \( c \) extending \( \bar{c} \) and meeting the dense sets \( \bar{D} \) in \( \omega_{I-1}+1 \) steps following this tower. Once we have defined \( N_i \), set \( \nu_i^m = \text{sup}(N_i \cap \omega_{N+1-m}) \).

Set
\[
N_0 = \text{the least } N < M \text{ such that } \omega_{I-2} \cup \{ \bar{c}, C, D \} \subseteq N;
\]
\[
N_{i+1} = \text{the least } N < M \text{ such that } N_i \cup \{ i, N_i \} \cup \{ E_m \cap \nu_i^m : 0 \leq m \leq N - I + 1 \} \subseteq N \text{ and there exists } \delta \in N \cap E_m \text{ such that } \delta > \text{sup}(E_m \cap N_i), \text{ for } 0 \leq m \leq N - I + 1; \text{ and } N_i = \bigcup_{j<i} N_j, \text{ if } i \text{ is a limit ordinal.}
\]

Note that if \( i < \omega_{I-1} \), then \( |N_i| = \omega_{I-2} \). Thus \( N_i \) and \( E_m \cap \nu_i^m \) lie in \( M \) and \( \text{sup}(E_m \cap N_i) < \beta^m \), for all \( m \). It follows that the recursive definition of \( N_i \) does not break down for \( i \leq \omega_{I-1} \).

Note that if \( i \leq \omega_{I-1} \), then
- \( N_i \prec H_\lambda \) and \( \bar{c}, C, \bar{D} \in N_i \);
- \( \nu_i^m = \text{sup}(N_i \cap \omega_{N+1-m}) \in E_m \), if \( i \) is a limit ordinal;
- \( \nu_i^\omega_{I-1} = \beta^m \); and
- \( \langle N_j : j \leq i \rangle \in N_{i+1}, \) if \( i < \omega_{I-1} \).

The last of these observations uses that \( \langle N_j : j \leq i \rangle \) is definable in \( N_{i+1} \) from the parameters \( N_i \) and \( E_m \cap \nu_i^m \), for \( 0 \leq m \leq N - I + 1 \).

Now we are prepared to construct the condition \( c \) extending \( \bar{c} \) and meeting the dense sets \( \bar{D} \). Define a descending sequence \( \langle c^i : i \leq \omega_{I-1} \rangle \) of conditions in \( C \) as follows.

[To make sense of the indexing, recall that
\[
C = \prod_{m=0}^{N-I+1} C(\omega_{I+m}, \omega_I, \omega_{I+m}^L),
\]
so \( c^i_{N-I+1-m} \in C(\omega_{N+1-m}, \omega_I, \omega_{N+1-m}^L) \). Also recall that the \( \beta^m \)’s and \( \lambda^m \)’s decrease as \( m \) increases. In particular, \( \omega_{N-m} < \beta^m < \omega_{N+1-m} \).]

Set \( c^0 = \bar{c} \). Choose \( c^{i+1} \) to be least in \( N_{i+1} \) such that
- \( c^{i+1} \subseteq c^i \) in \( C \);
- \( c^{i+1}_{N-I+1-m} \setminus c^i_{N-I+1-m} \) contains an ordinal greater than \( \nu_i^m \); and
- \( c^{i+1} \) meets \( D_i \).

If \( i \leq \omega_{I-1} \) is a limit ordinal, set
\[
c^i_{N-I+1-m} = \bigcup_{j<i} c^j_{N-I+1-m} \cup \{ \nu_i^m \}
\]
and set \( c^i = (c_0^i, \ldots, c_{N-I+1}^i) \).

Proceed by induction on \( i \) to see that \( c^i \in \mathcal{C} \) and that if \( i < \omega_{I-1} \), then \( c^i \in N_{i+1} \).

If \( i \) is a successor ordinal, inductively it suffices to observe that \( i \), hence \( D_i \), lies in \( N_{i+1} \). If \( i < \omega_{I-1} \) is a limit ordinal, then \( c^i \) is definable in \( N_{i+1} \) from the parameters \( \{ N_j : j \leq i \} \), \( c \), \( \mathcal{C} \), \( \bar{D} \), and \( i \). Hence \( c^i \in N_{i+1} \).

If \( i < \omega_{I-1} \) is a limit ordinal, then \( \sup(\bigcup_{j<i} c_{N-I+1-m}^j) = \nu_i^m \) by construction. And \( \nu_i^m \in E_m \). Consequently, if \( \text{cf}(\nu_i^m) = \omega_1 \), then \( \text{cf}^{L}(\nu_i^m) = \omega_{N+1-m}^{L} \). Hence \( c_{N-I+1-m}^i \in \mathcal{C}(\omega_{N+1-m}, \omega_I, \omega_{N+1-m}^{L}) \).

Finally, \( \sup(\bigcup_{j<\omega_{I-1}} c_{N-I+1-m}^j) = \beta^m \). And \( \text{cf}(\beta^m) = \omega_{I-1} \). If \( I - 1 > 1 \), it follows that \( c_{\omega_{I-1}}^i \in \mathcal{C} \). If not, then \( I = 2 \), since \( I > 1 \). In this case, \( \text{cf}^{L}(\beta^m) = \lambda^m \) and \( \lambda^m = \omega_{N+1}^{L} = \omega_{N-I-m}^{L} \). So \( c_{N-I+1-m}^{\omega_{I-1}} \in \mathcal{C}(\omega_{N+1-m}, \omega_I, \omega_{N+1-m}^{L}) \), as required.

The condition \( c = c_{\omega_{I-1}} \) extends \( \bar{c} \) in \( \mathcal{C} \) and meets the dense sets \( \bar{D} \).